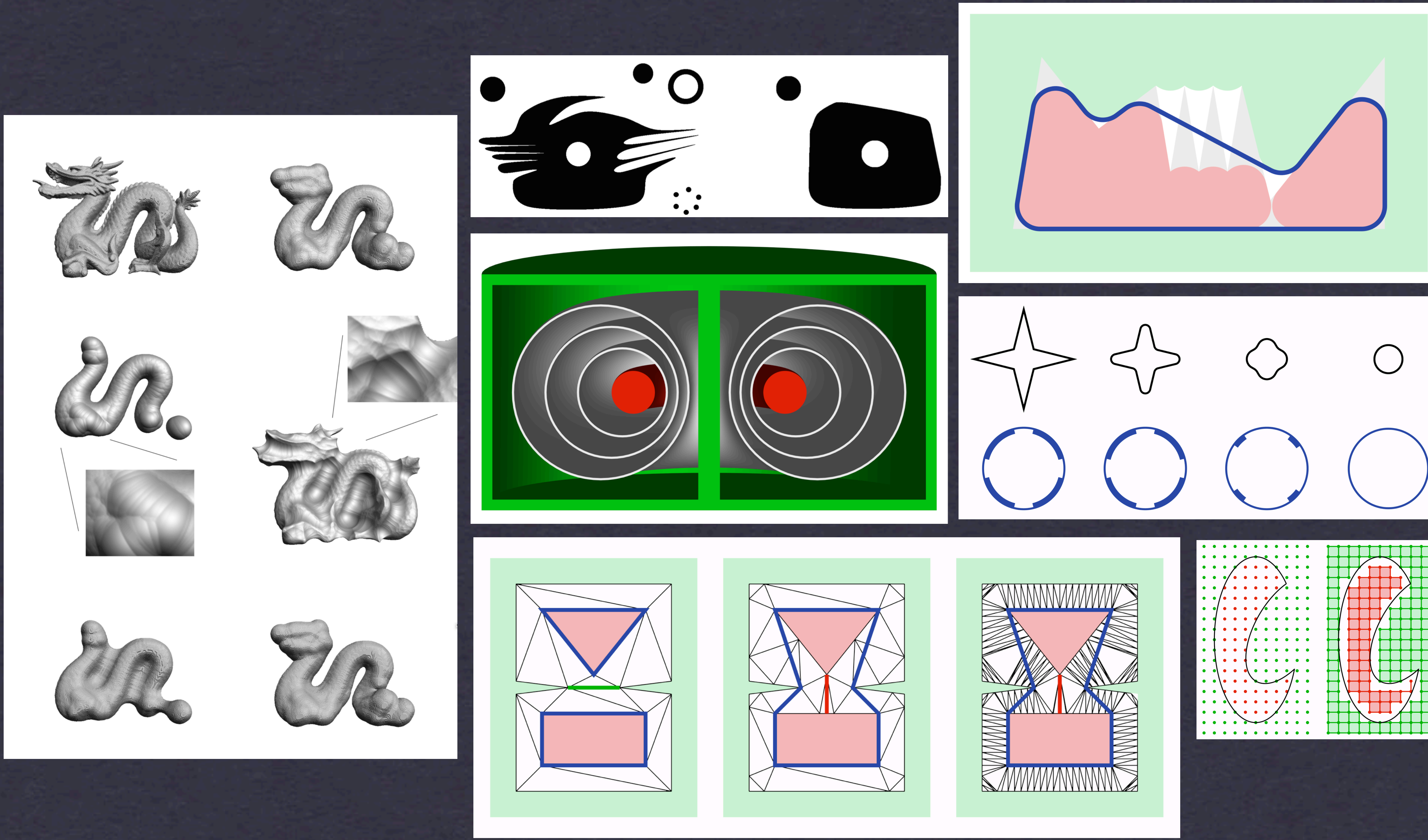
Tightening and Blending

Subject to Set-Theoretic Constraints

Jason Williams



EXAMPLES

MASON, TIGHTENING, TIGHT HULL, TIGHT BLEND, MEDIAL COVER

Problems

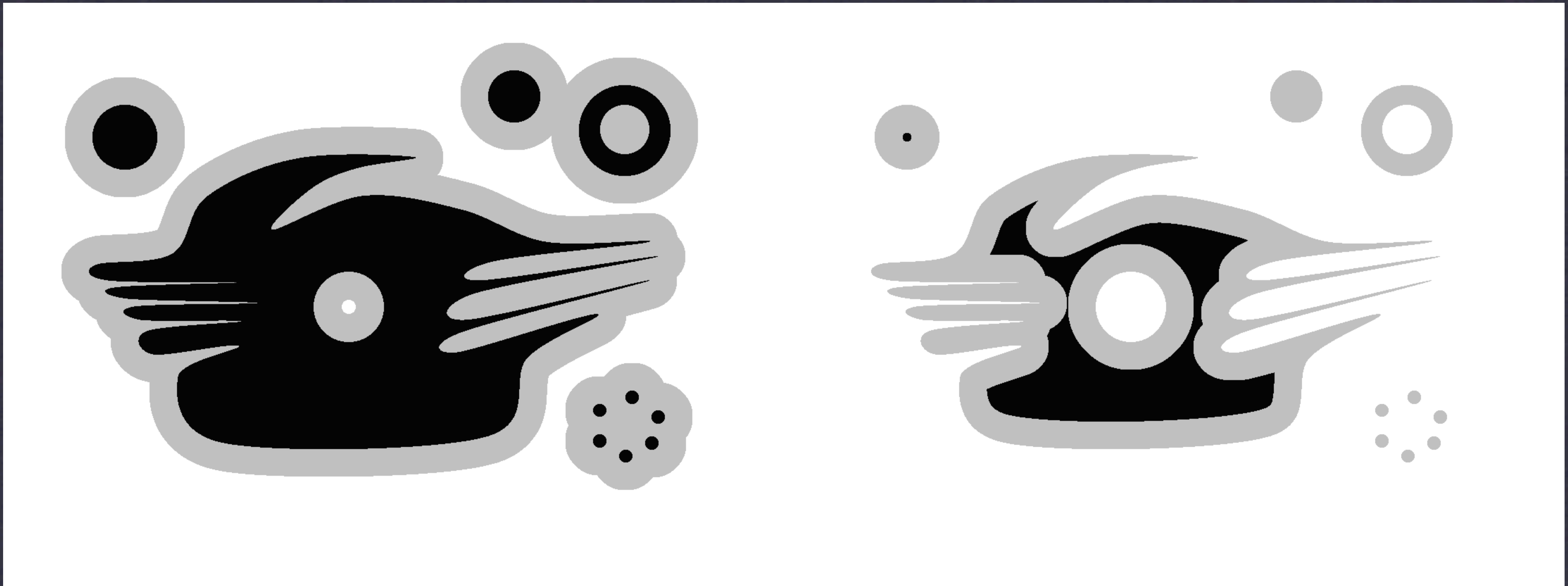
- Blend and tighten solid input shapes.
  - Blending smooths thin, sharp features.
  - Tightening reduces boundary measure.
- Output contains one set and excludes a second.
- Optimize the separating boundary.

Techniques

- Mason: symmetrically regularize
- Tightening: bound mean curvature with flow
- Tight Hull: symmetric convex hull generalization
- Tight Blend: tight hull normal field simplification
- Medial Cover: medial hull and blend topology

Mathematical Morphology

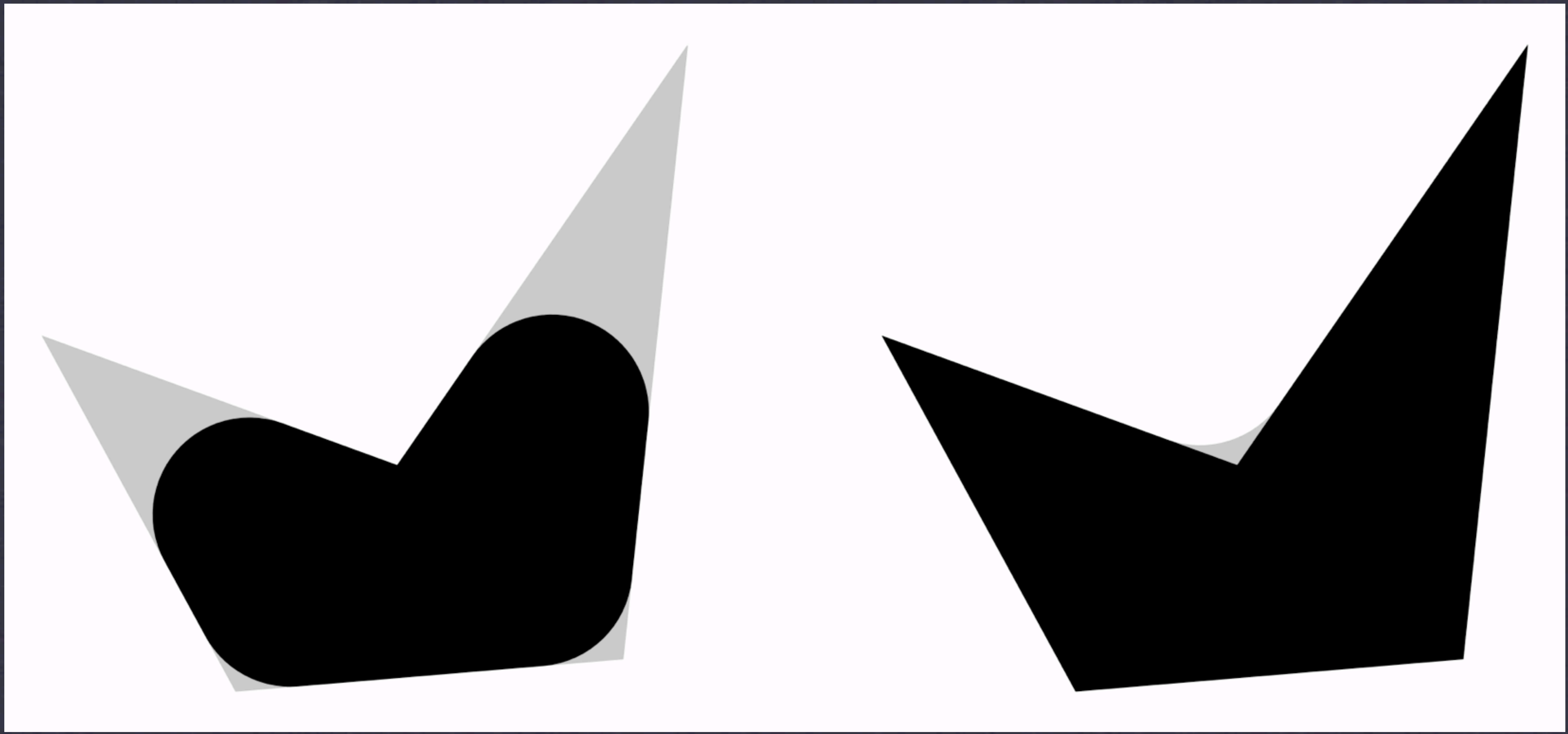
- Set-theoretic operations on sets of balls
- Scale dependence, parameterized by radius
- Growing and shrinking
- Opening and closing
- Mortar, core, and anticore
- Regular sets and regularity



Growing and Shrinking

$$S\uparrow_r := \bigcup \mathbb{B}_r\left(S\right)$$

$$S\downarrow_r := S^c\uparrow_r{}^c$$

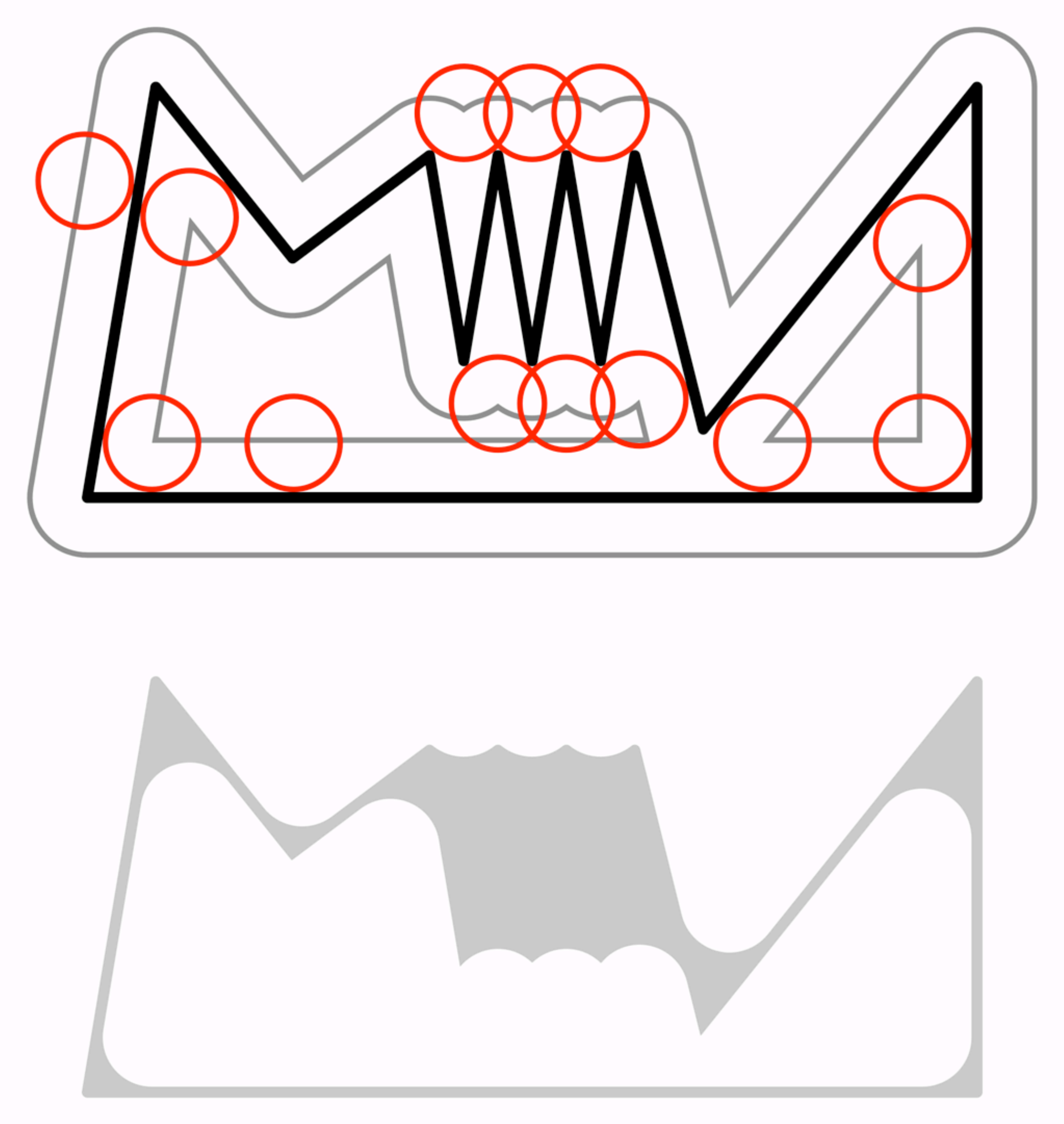


Morphological Opening and Closing

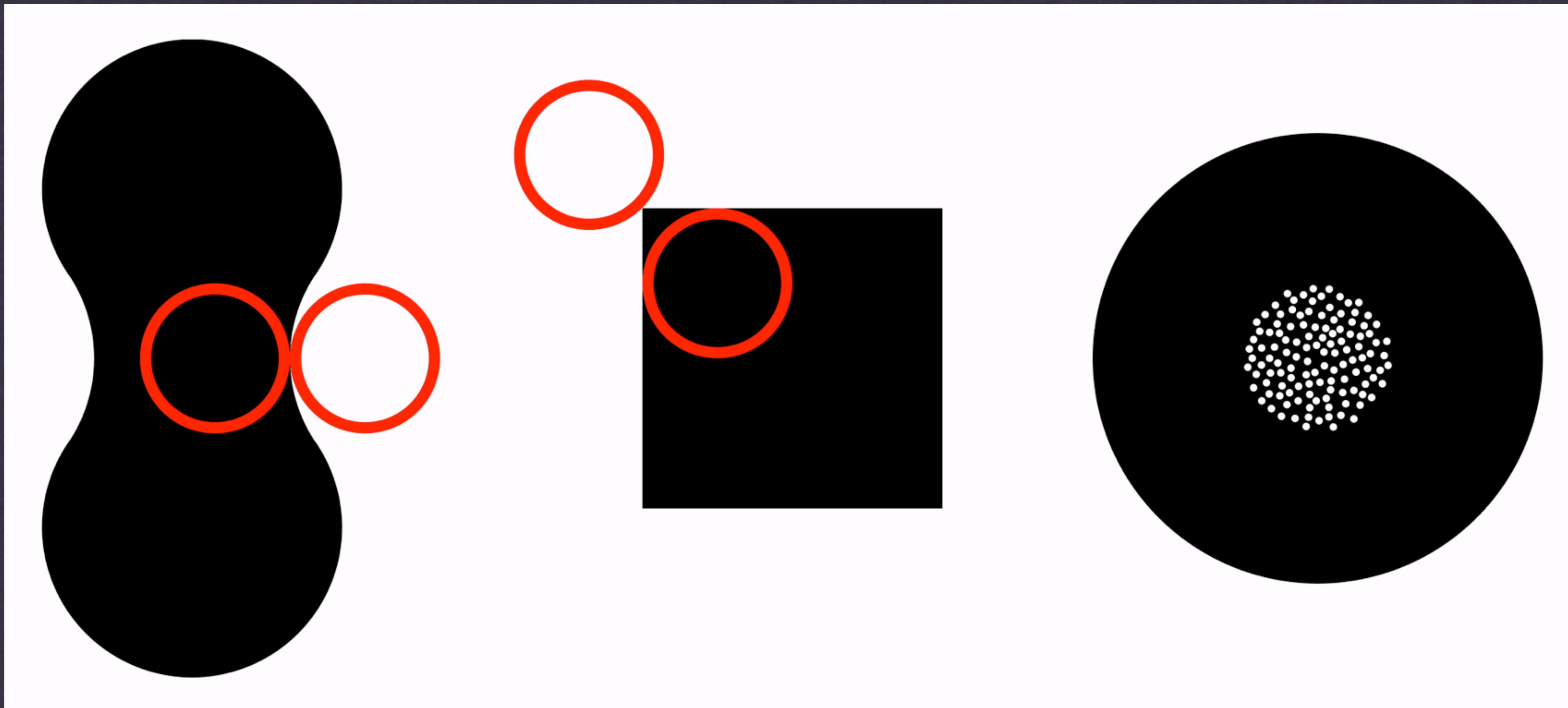
$$S \circ_r := \bigcup \mathbb{B}_r^S$$

$$S \bullet_r := {S^c \circ_r}^c$$

Mortar, Core, Anticore



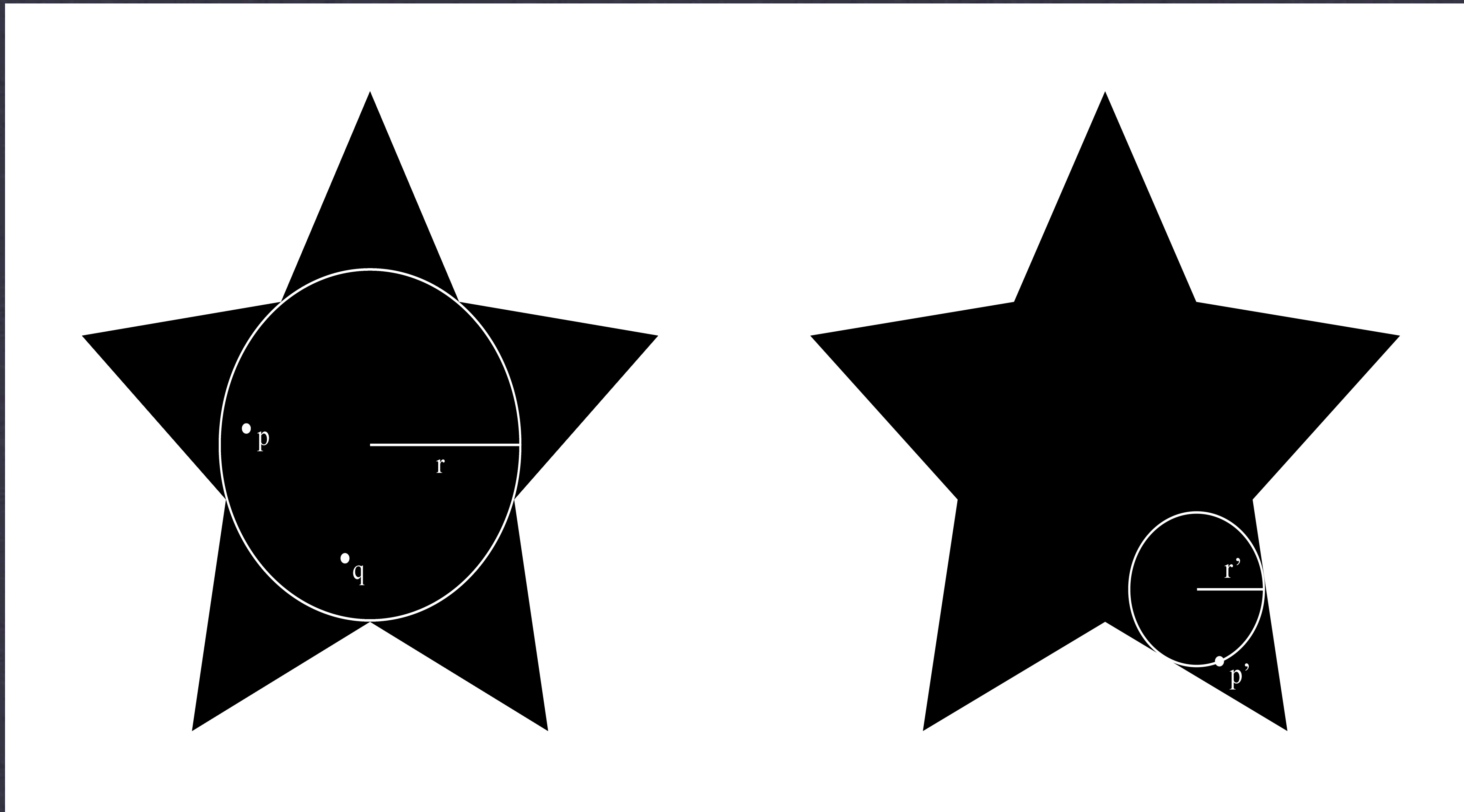
$$\mathcal{M}_r S := S \bullet_r \cap S^c \bullet_r$$



$$S = S \circ_r \bullet_r$$

REGULAR SET

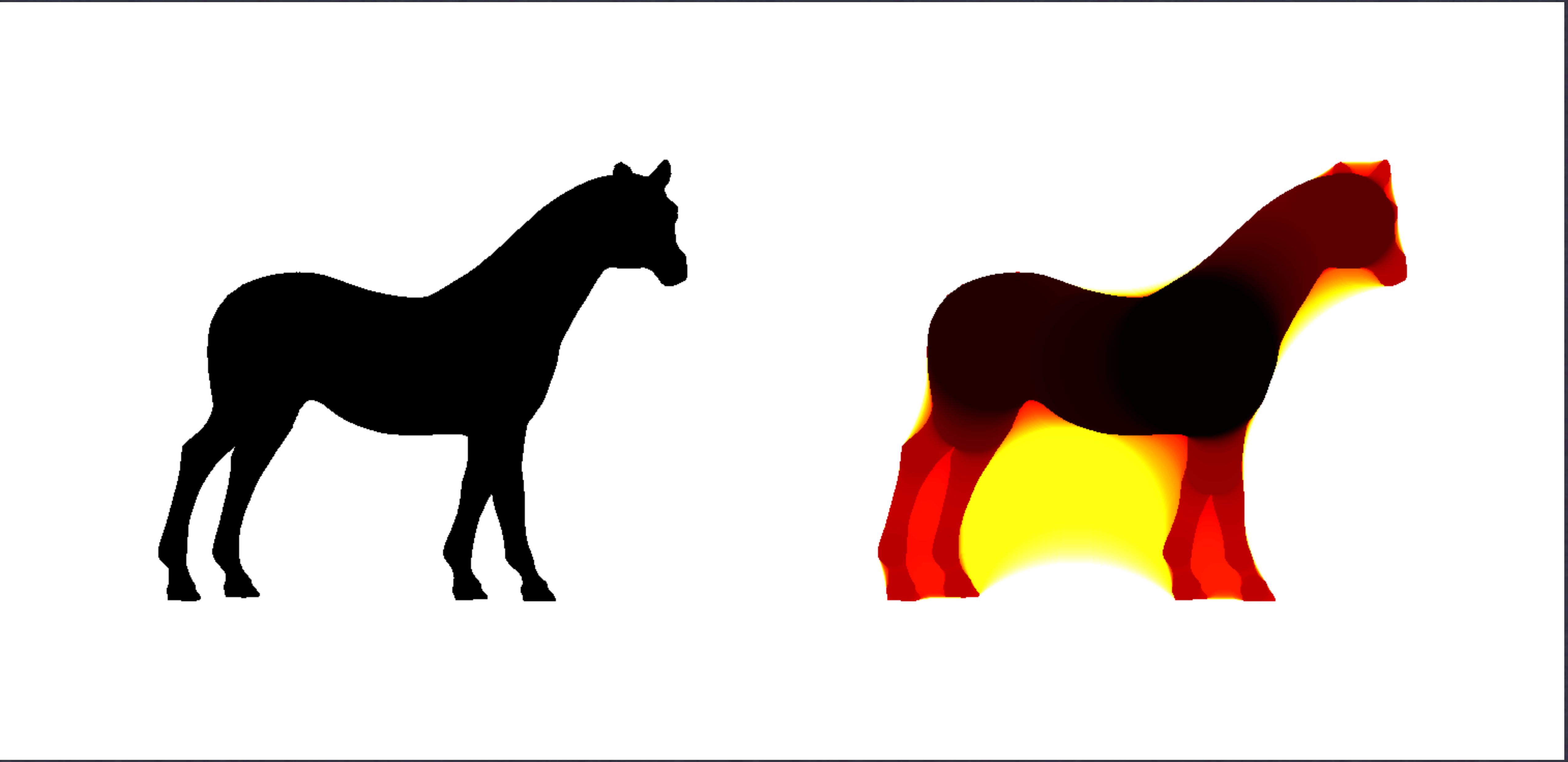
A CLOSED SET S WITH MANIFOLD BOUNDARY IS REGULAR IF AND ONLY IF...



$$\mathcal{R}^{S}\left(p\right) := \bigvee \left\{\left\{r \in \mathbb{R}^{>} \mid p \in S \circ_{r}{}^{-}\right\} \cup 0\right\}$$

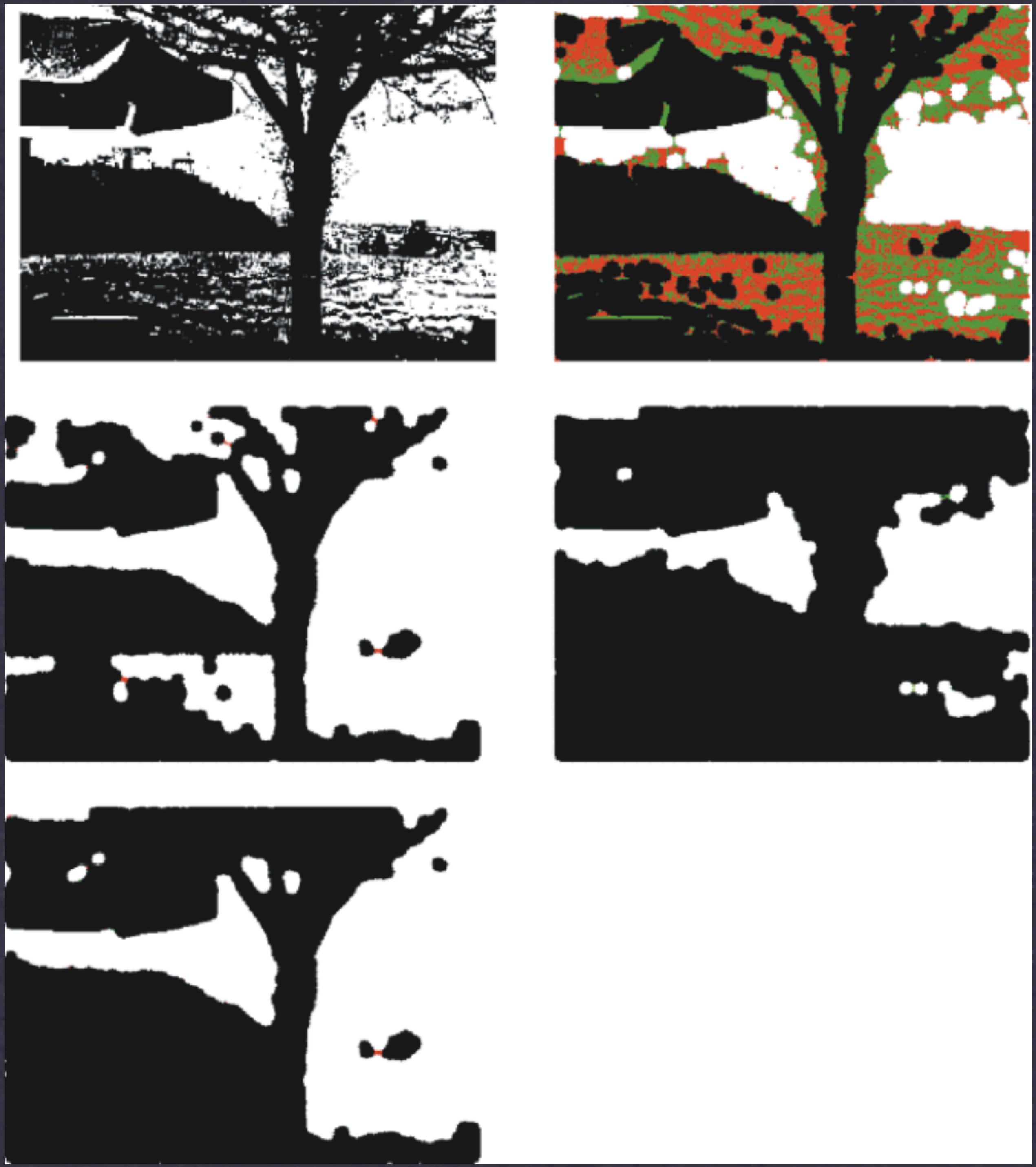
REGULARITY

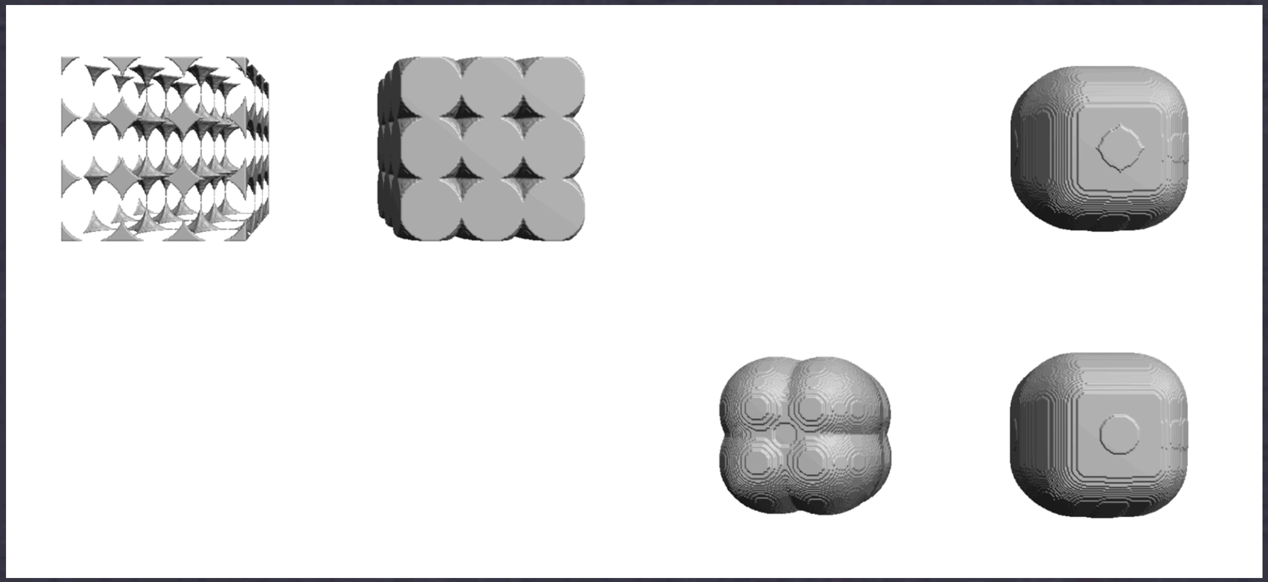
POINTS P AND Q ARE R-REGULAR, WHILE P' IS R'-REGULAR.



REGULARITY TRANSFORM

A HORSE ON THE LEFT, WITH PIXELS ON THE RIGHT COLORED BY REGULARITY.





MASON

SYMMETRIC REGULARIZATION THROUGH MORPHOLOGICAL OPENING AND CLOSING

Mason Definition

$$T_r S \cap (\mathcal{M}_r S)^c = (\mathcal{M}_r S)^c$$

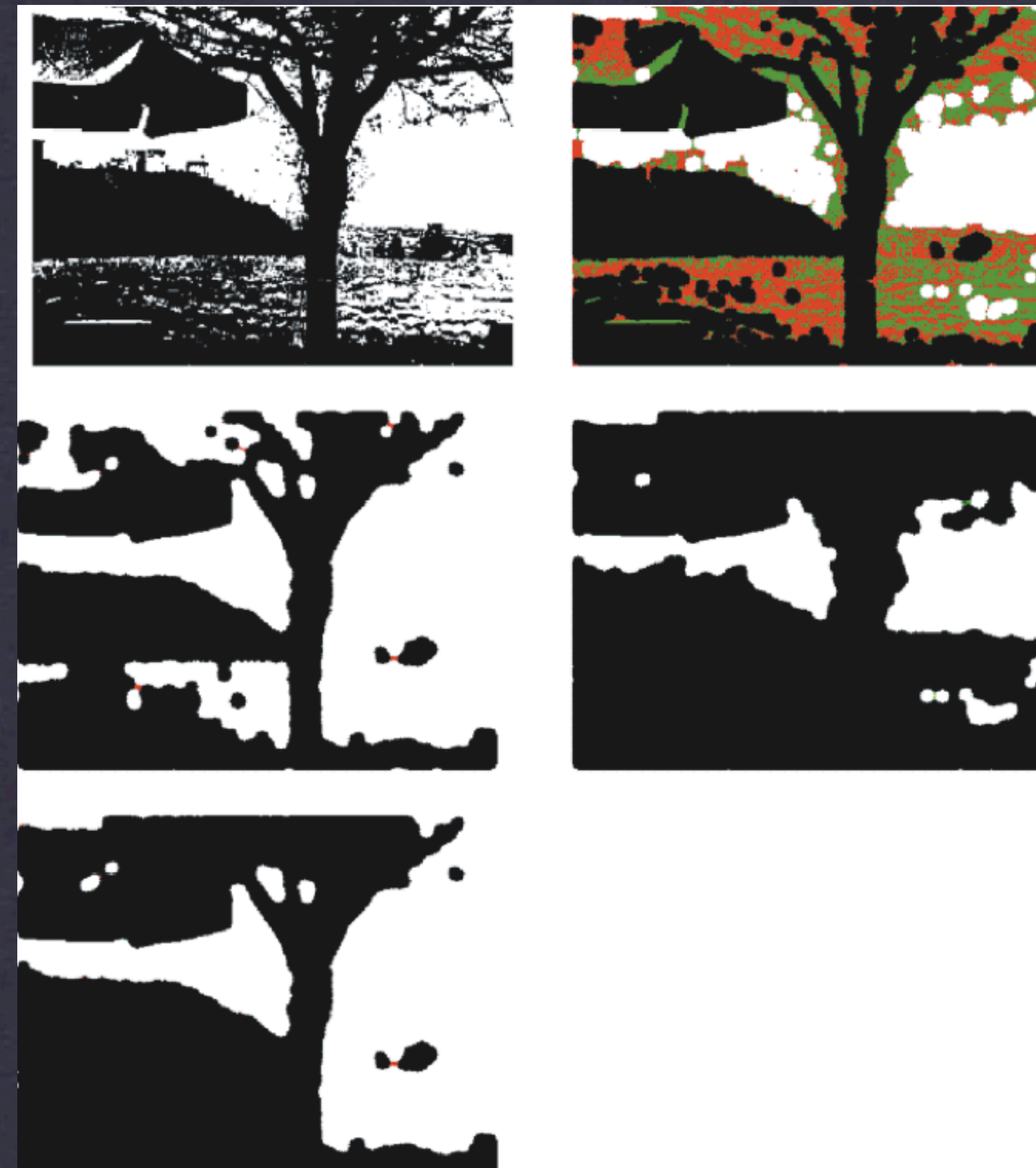
$$C_r S := \left\{\rho^- \mid \rho \in \kappa\left((\mathcal{M}_r S)^\circ\right)\right\}$$

$$T_r S \cap C_r^i S = \begin{cases} C_r^i\left(S \circ_r \bullet_r\right) & \mu\left(C_r^i S \setminus C_r^i\left(S \circ_r \bullet_r\right)\right) \leq \mu\left(C_r^i S \setminus C_r^i\left(S \bullet_r \circ_r\right)\right) \\ C_r^i\left(S \bullet_r \circ_r\right) & \mu\left(C_r^i S \setminus C_r^i\left(S \bullet_r \circ_r\right)\right) < \mu\left(C_r^i S \setminus C_r^i\left(S \circ_r \bullet_r\right)\right) \end{cases}$$

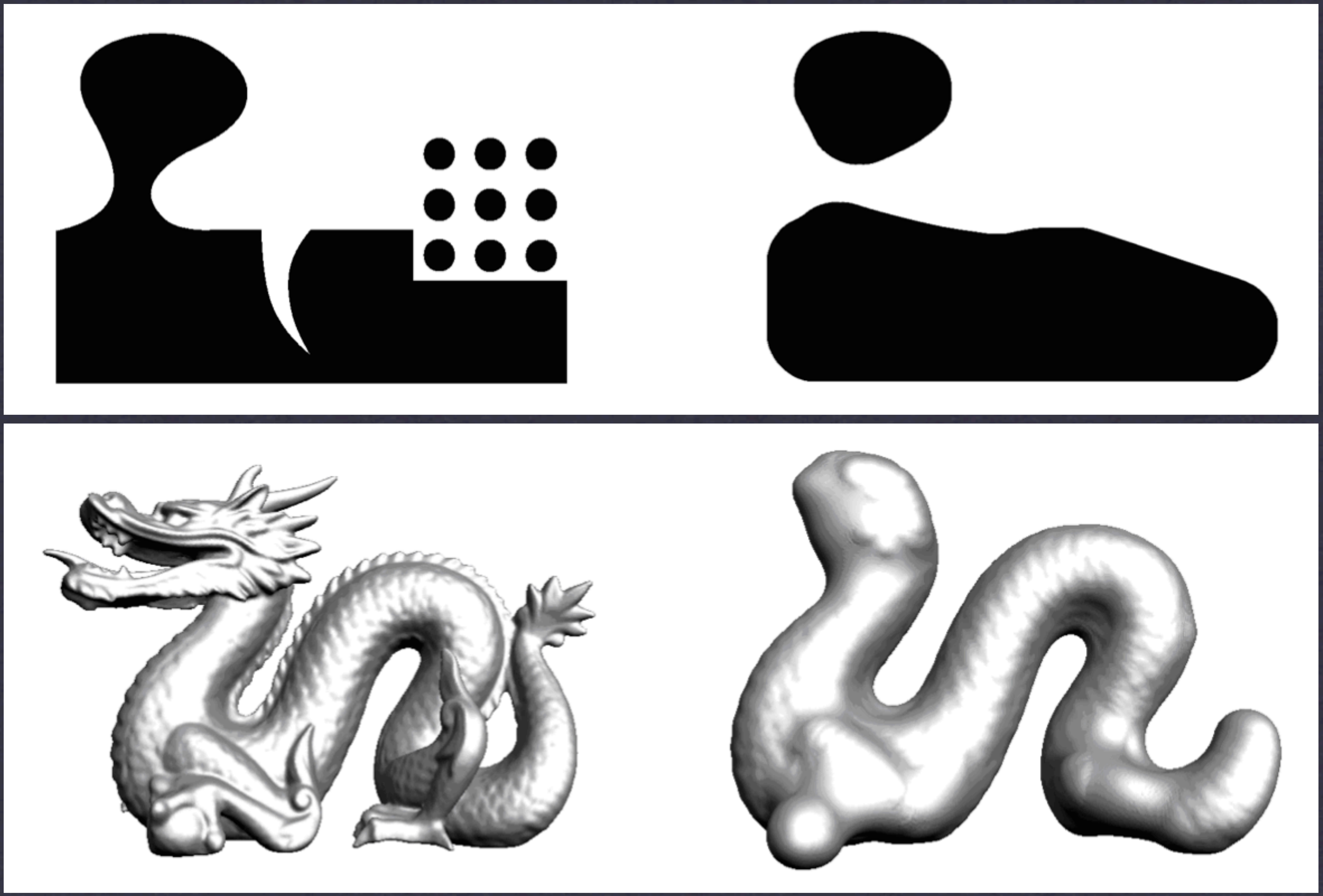
- Mason equals the input outside the mortar of the input.
- In a component of the input's mortar, mason chooses open/close or close/open to minimize set difference.

Minimizing Symmetric Difference

Mortar (bottom left) yields a smaller area change than open/close (middle left) or close/open (middle right.)



	Area	Change	Black	White	Inner	Outer
S	78,195	N/A	527	711	18,676	18,641
$S \circ_5 \bullet_5$	62,808	19,301	6	23	107	0
$S \bullet_5 \circ_5$	95,472	18,573	1	33	0	43
$\mathcal{T}_5 S$	75,674	15,071	3	27	48	13



TIGHTENING

MINIMIZING BOUNDARY MEASURE AND MEAN CURVATURE THROUGH MEAN CURVATURE FLOW

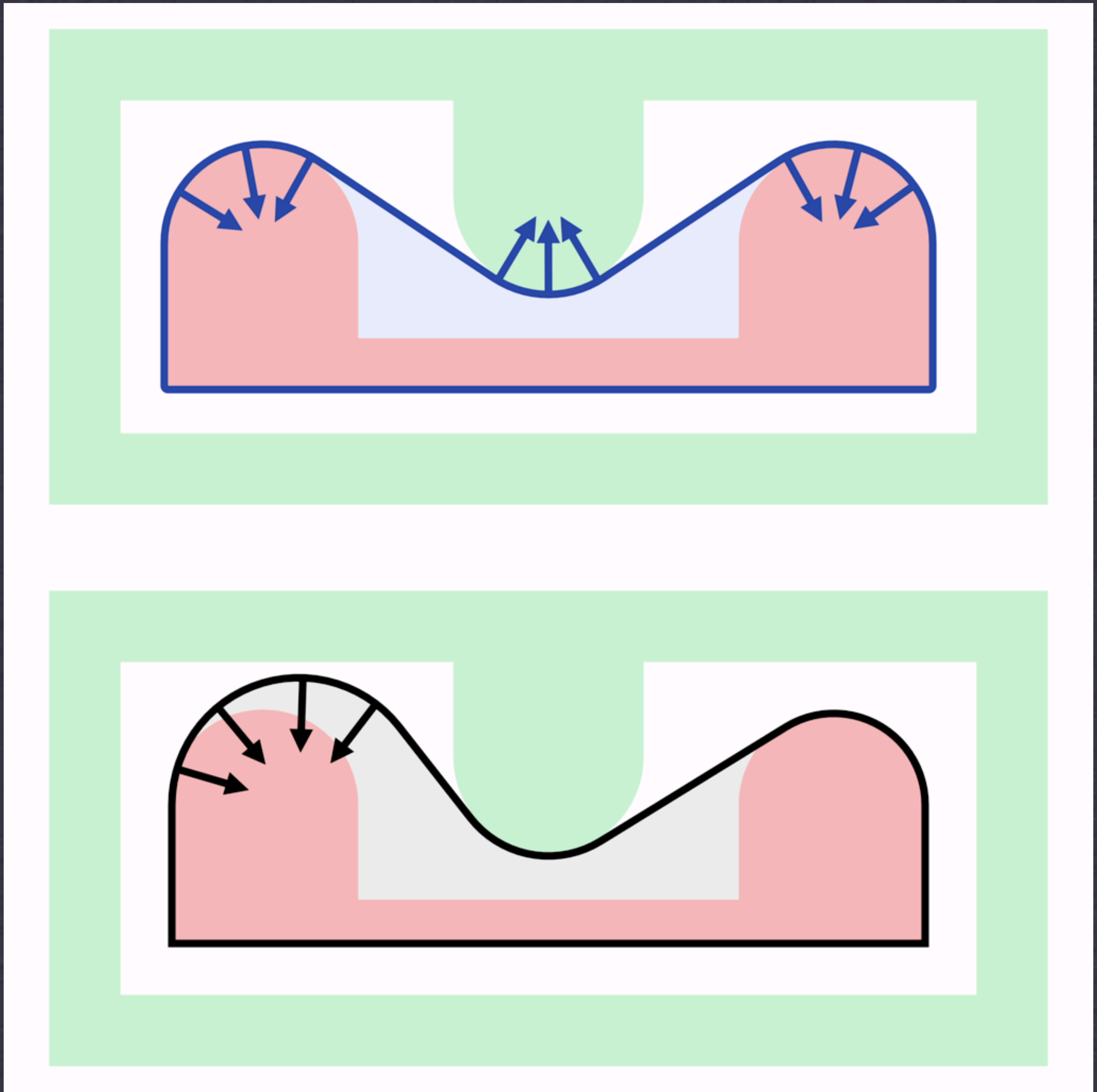
Tightening Definition

A set $C \subseteq \mathbb{R}^d$ is a *candidate r-tightening* of $S \subseteq \mathbb{R}^d$ if and only if $S \circ_r \subseteq C$ and $S \bullet_r{}^c \cap C = \emptyset$.

A candidate r-tightening set $T \subseteq \mathbb{R}^d$ is an *r-tightening* of $S \subseteq \mathbb{R}^d$ if and only if there is no candidate r-tightening set $C \subseteq \mathbb{R}^d$ such that $\mathrm{H}^{d-1}(C) < \mathrm{H}^{d-1}(T)$ given that C is isotopic to T throughout an isotopy in which C continuously remains a candidate r-tightening.

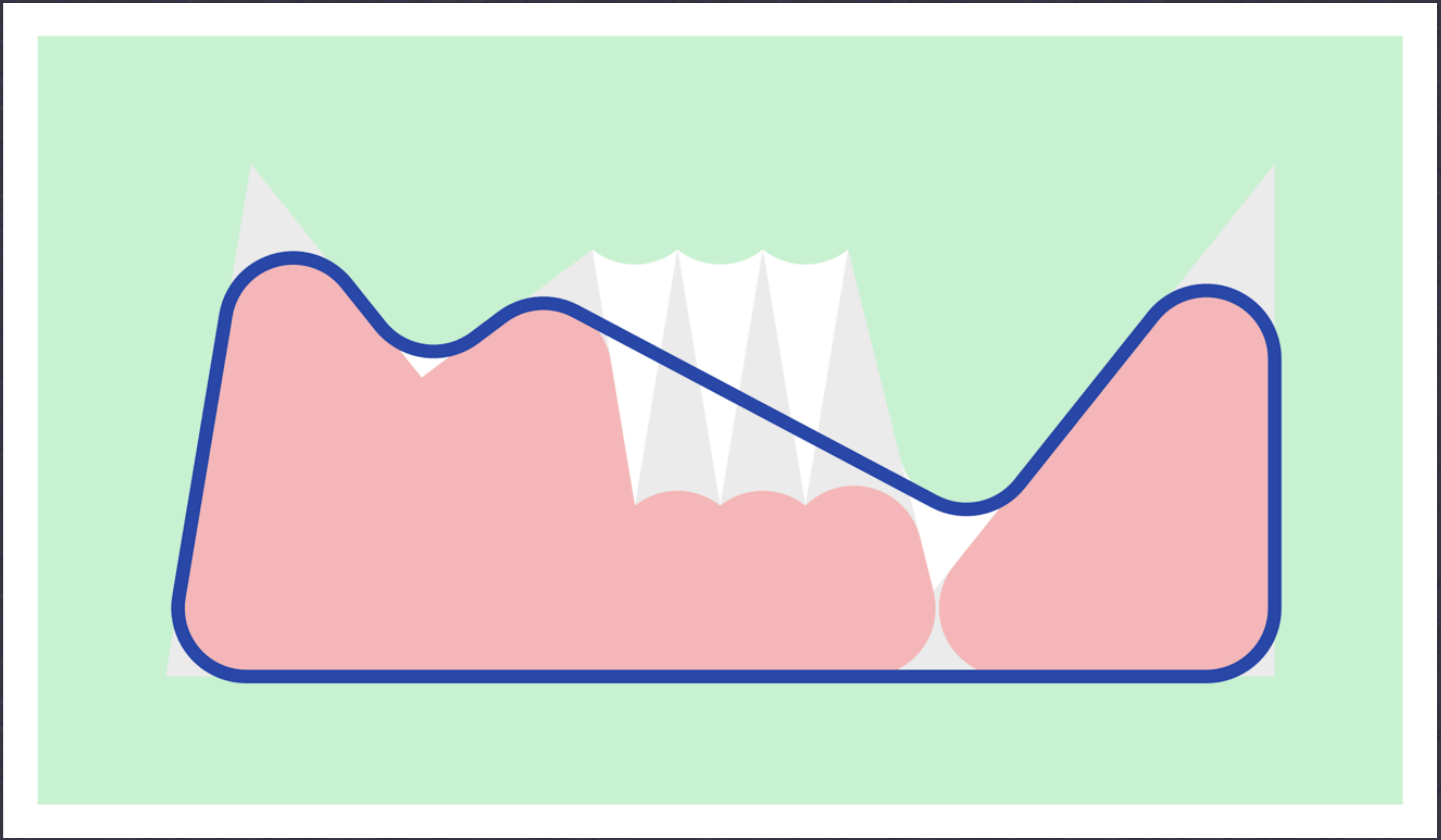
Mean Curvature Flow

- Mean curvature at a smooth boundary point is the mean of the principal curvatures at that point.
- A mean curvature normal scales a unit normal at a boundary point by the mean curvature at that point.
- Mean curvature flow moves a boundary point along its mean curvature normal.
- Mean curvature flow is along the gradient of boundary measure, stable against constraints.
- Unconstrained boundary is stable where mean curvature is zero.



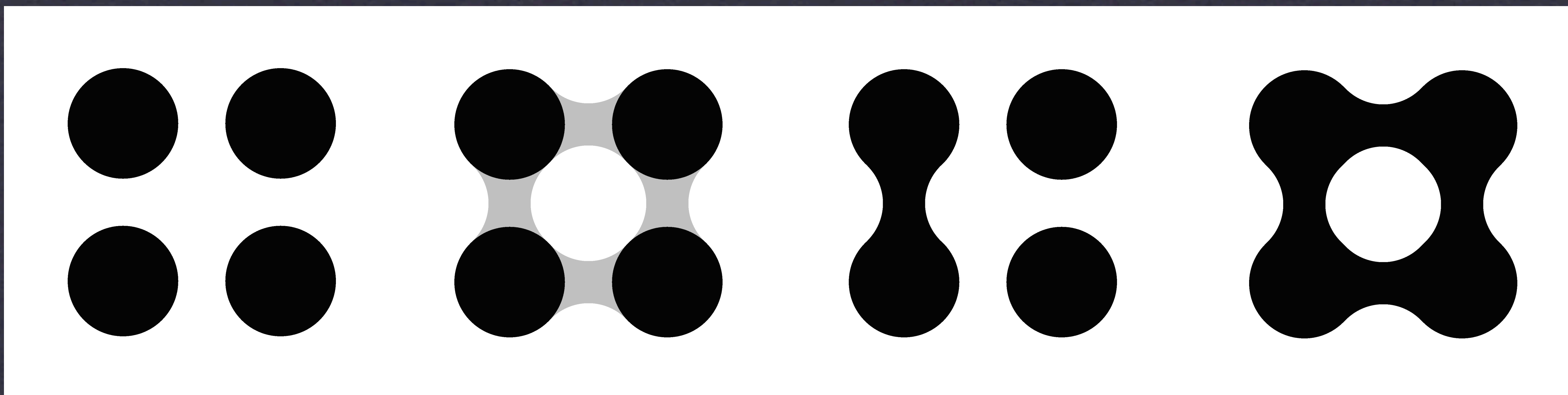
MEAN CURVATURE FLOW STABILITY

FLOW IS STABLE IF AND ONLY IF NORMALS POINT INWARD AT CONSTRAINT BOUNDARIES.



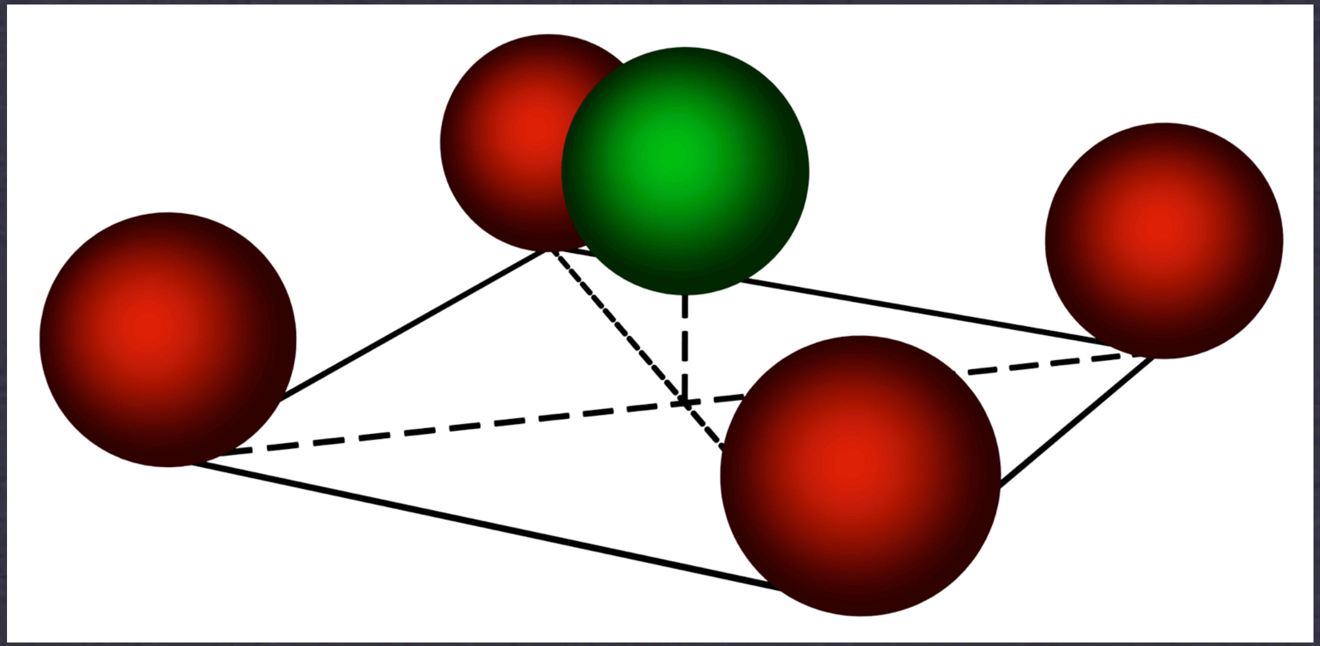
MEAN CURVATURE BOUND

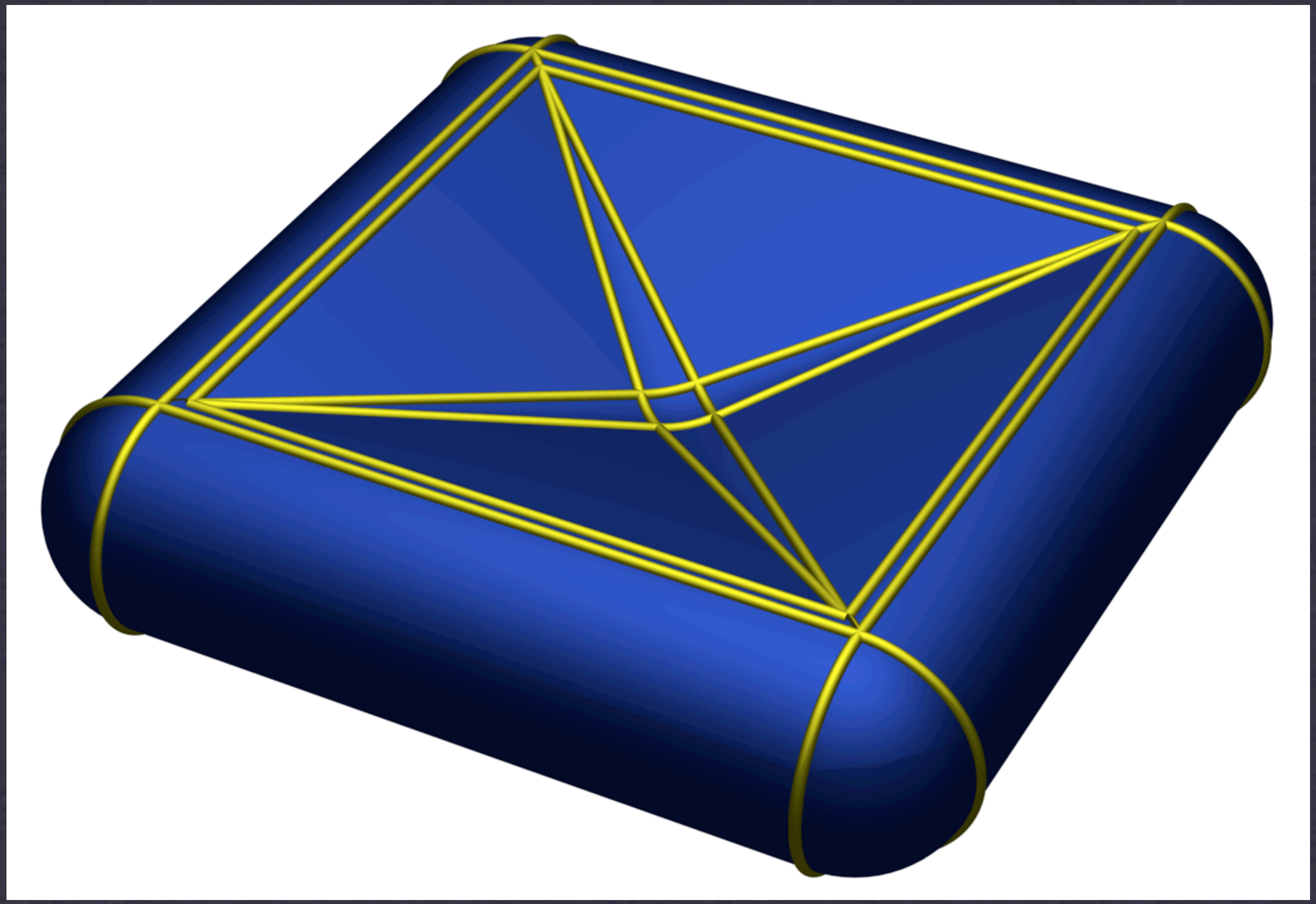
R-TIGHTENING MEAN CURVATURE HAS MAGNITUDE LESS THAN OR EQUAL TO 1/R.



MULTIPLE R-TIGHTENING TOPOLOGIES

MULTIPLE TIGHTENING BOUNDARIES LOCALLY MINIMIZE BOUNDARY MEASURE.





TIGHT HULL

SYMMETRIC TIGHT HULL GENERALIZATION MINIMIZING SLACK AND BOUNDARY

Convex Hull

A set $S \subseteq \mathbb{R}^d$ is *convex* if and only if for every two points $p, q \in S$, the closed line segment $\overline{pq}$ is contained in S.

The *convex hull* of $S \subseteq \mathbb{R}^d$ is the intersection of all convex sets containing S.

Relative Convex Hull

For $G \subseteq \mathbb{R}^d$, a set $S \subseteq \mathbb{R}^d$ disjoint from G is *convex relative to* G if and only if for every two points $p, q \in S$ such that $\overline{pq}$ is disjoint from G, set S contains $\overline{pq}$.

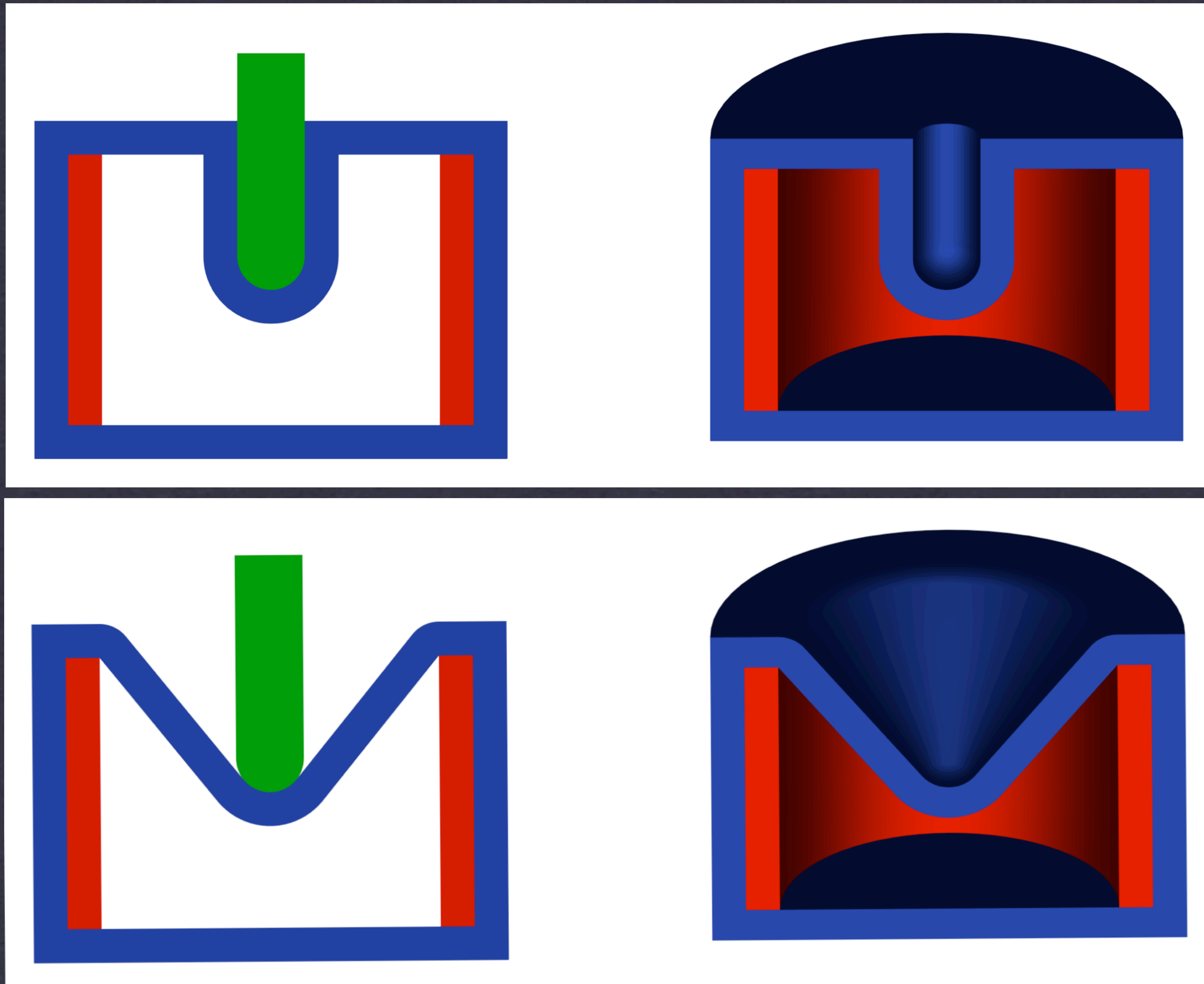
For $R \subseteq \mathbb{R}^d$ disjoint from $G \subseteq \mathbb{R}^d$, the *convex hull of* R *relative to* G is the intersection of all sets convex relative to G that contain R.

Tight Hull Definition

For $R, G \subseteq \mathbb{R}^d$, suppose $R \cap G = \emptyset$. Then $C \subseteq \mathbb{R}^d$ is a *candidate tight hull* of R relative to G if and only if

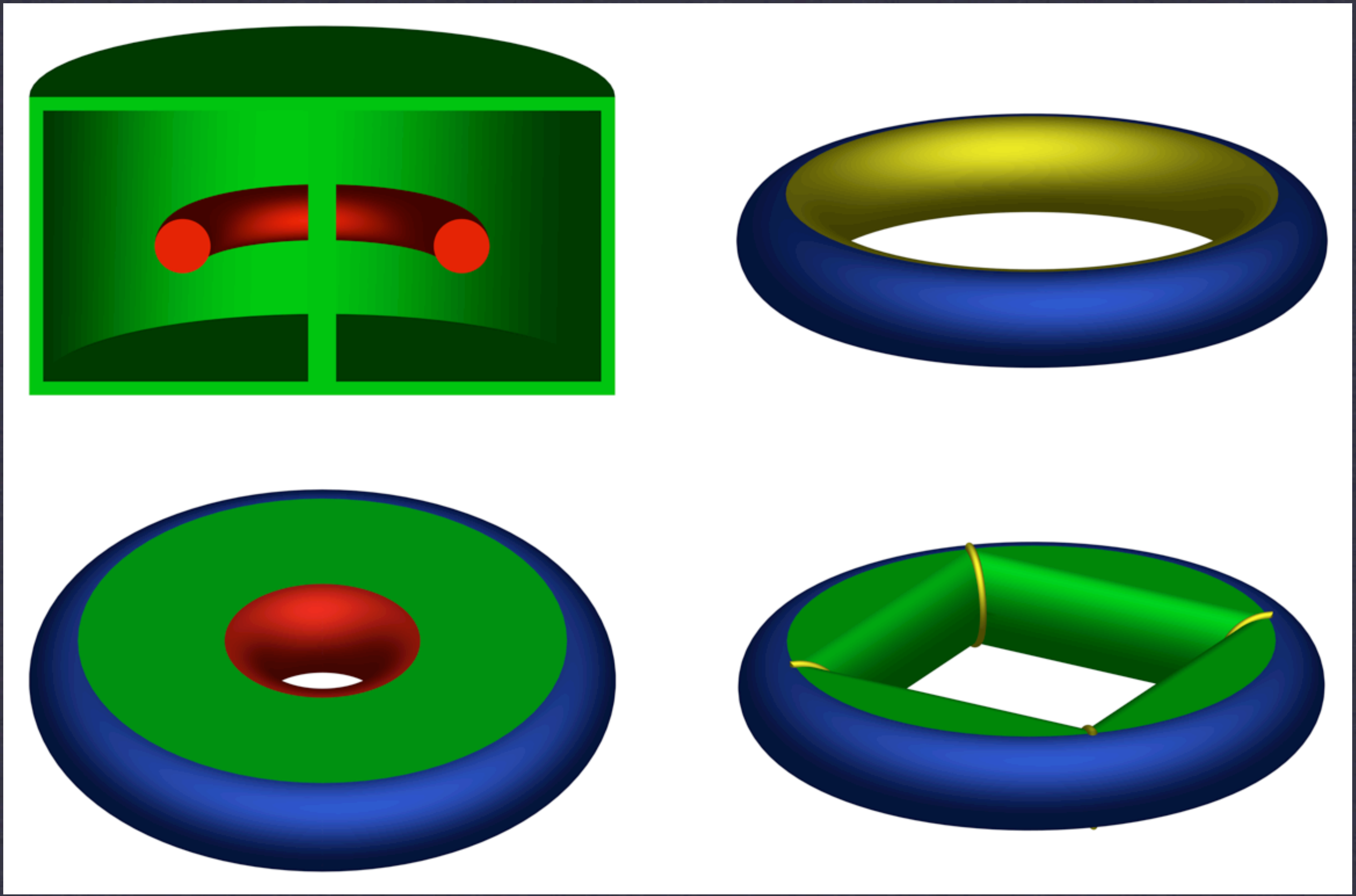
1. $R \subseteq C$
2. $C \subseteq G^c$
3. There is no $S \subseteq \mathbb{R}^d$ such that $R \subseteq S$, $S \subseteq G^c$, and the slack of S is less than the slack of C.

A candidate tight hull T of R relative to G is a *tight hull* of R relative to G if and only if there is no candidate tight hull S of R relative to G such that the unsupported slack of S is less than the unsupported slack of T.



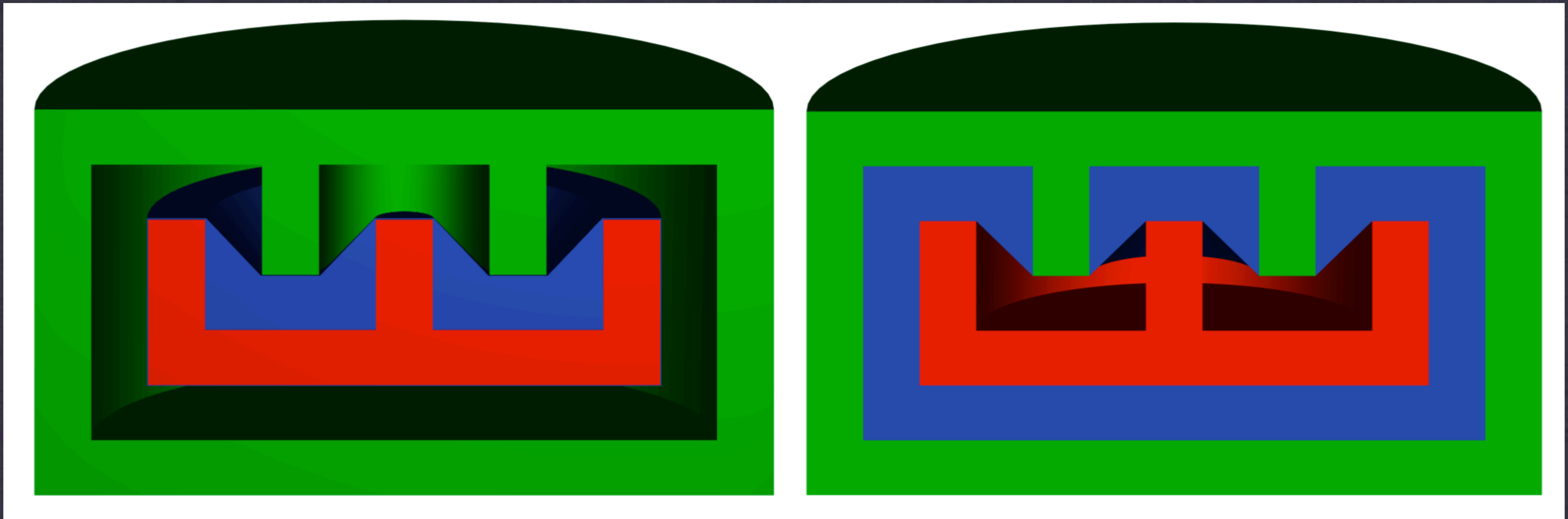
RELATIVE CONVEX HULL AND TIGHT HULL

FLUID BOUNDARY ABOVE, MEMBRANE IN TENSION BELOW



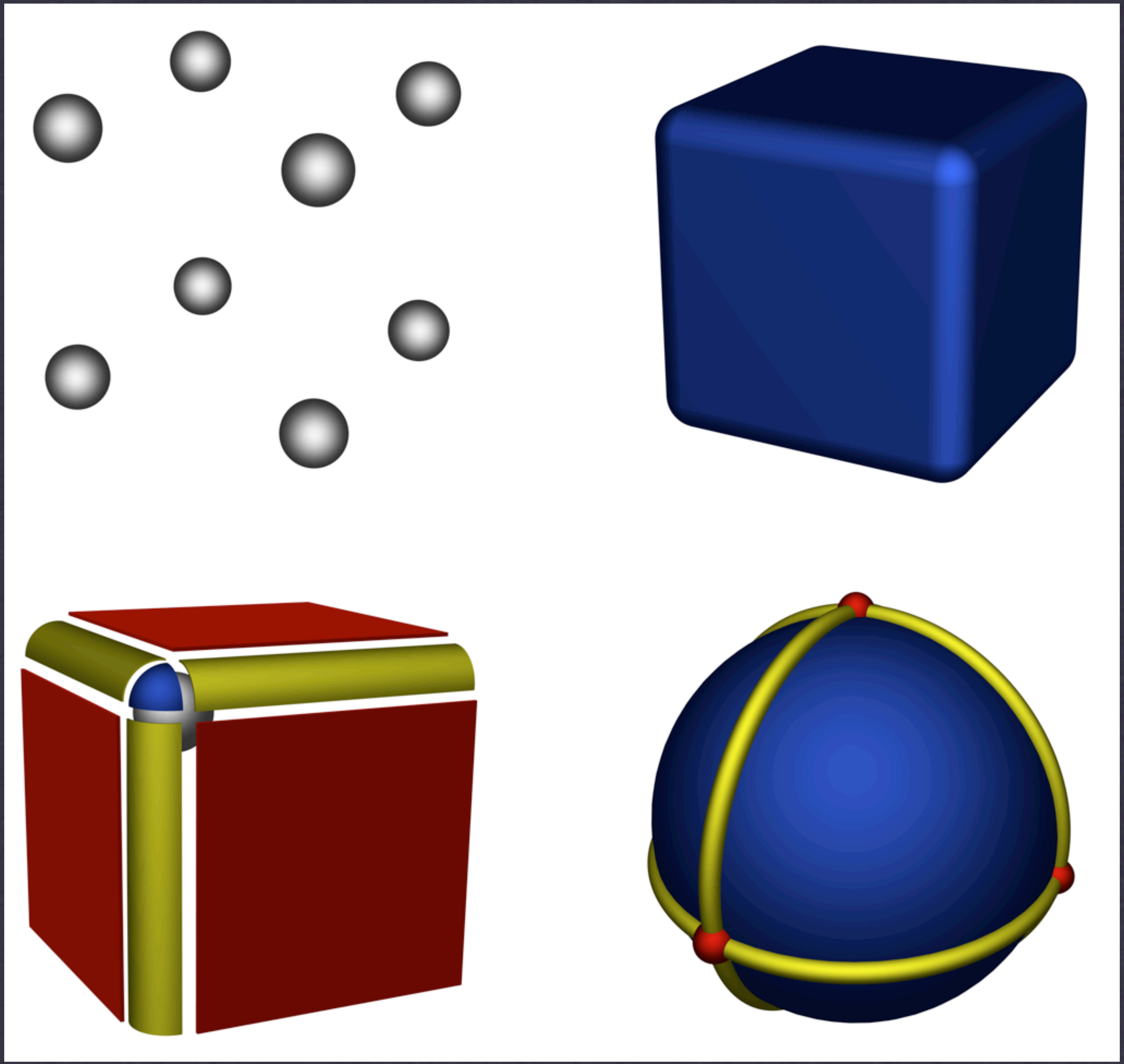
MINIMIZING UNSUPPORTED SLACK

SUPPORTED (BLUE) PARTIALLY (YELLOW) UNSUPPORTED (RED) DEVELOPABLE (GREEN)



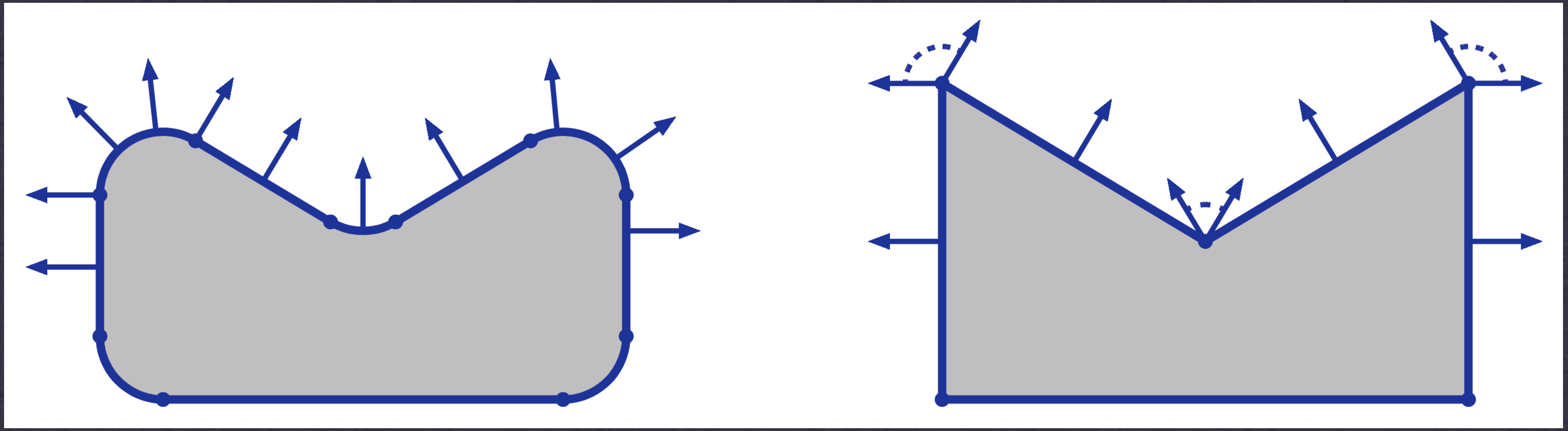
SYMMETRIC HULL

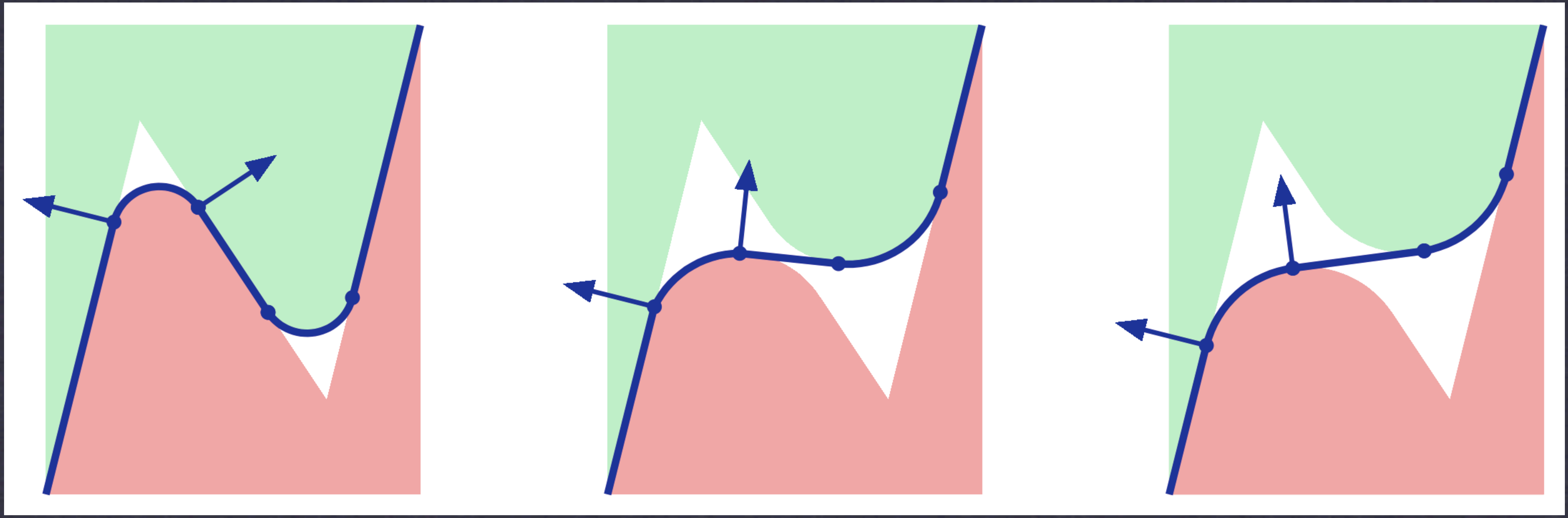
THE TIGHT HULL OF R RELATIVE TO G AND THE TIGHT HULL OF G RELATIVE TO R



DEVELOPABLE BOUNDARY

THE CONVEX HULL'S CYLINDRICAL AND SQUARE BOUNDARY PATCHES ARE DEVELOPABLE.





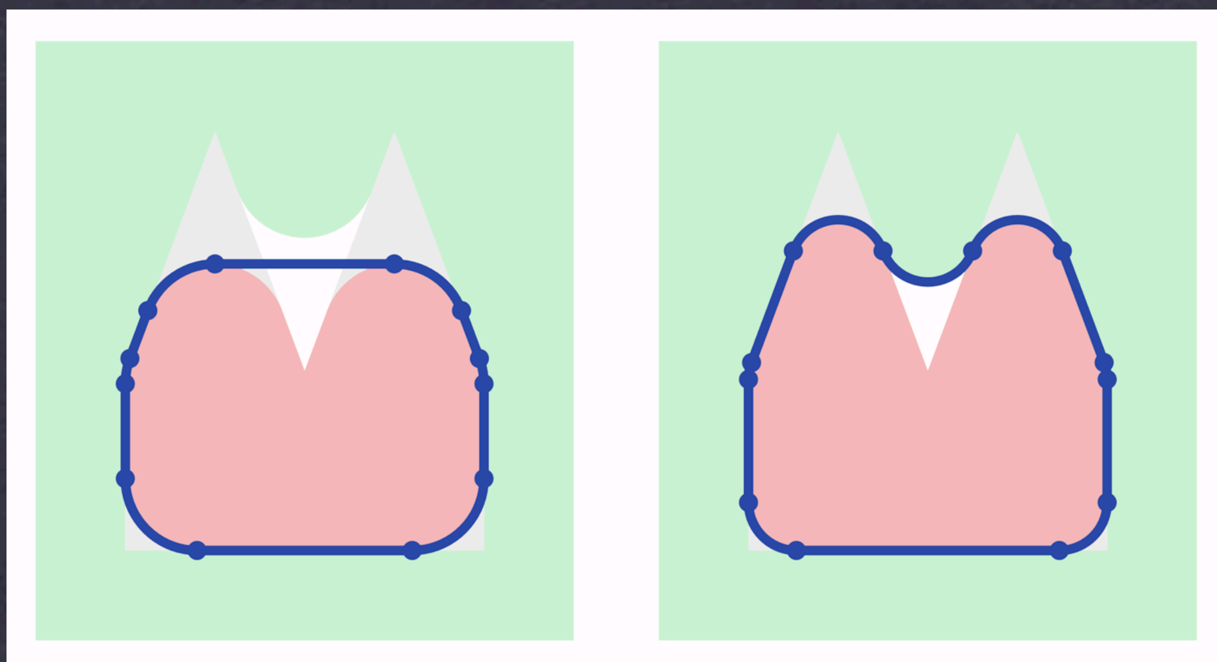
TIGHT BLEND

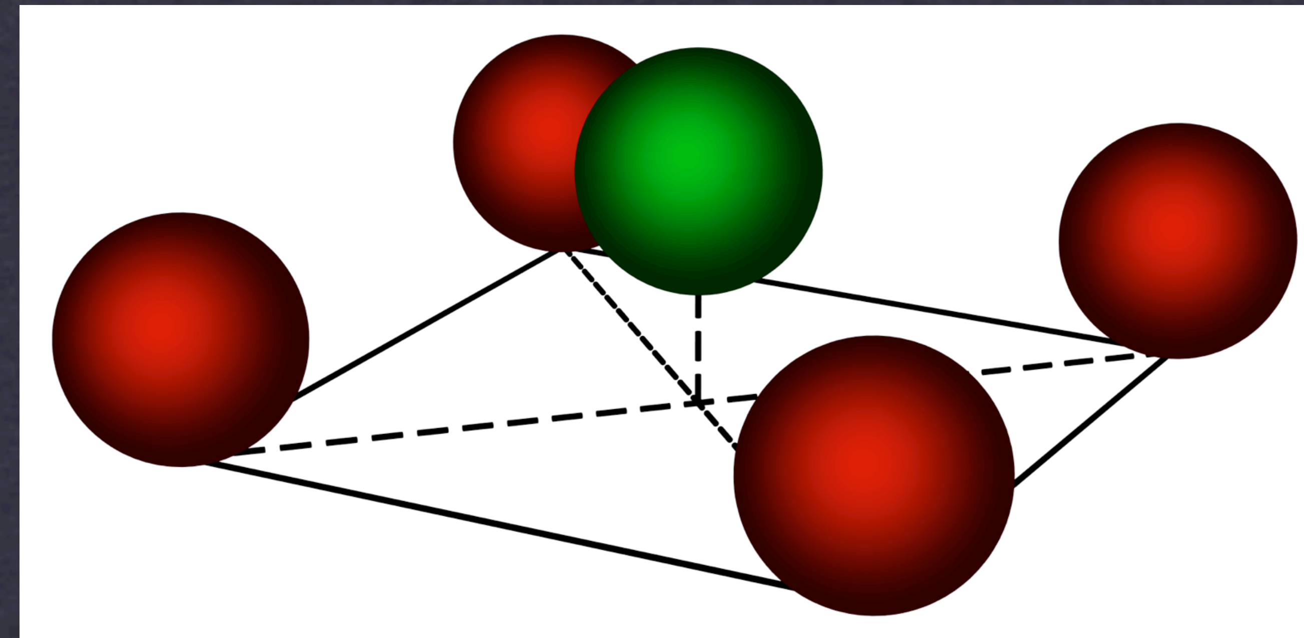
OPENING INPUT INTERIOR AND COMPLEMENT PROGRESSIVELY SIMPLIFIES NORMAL FIELD.

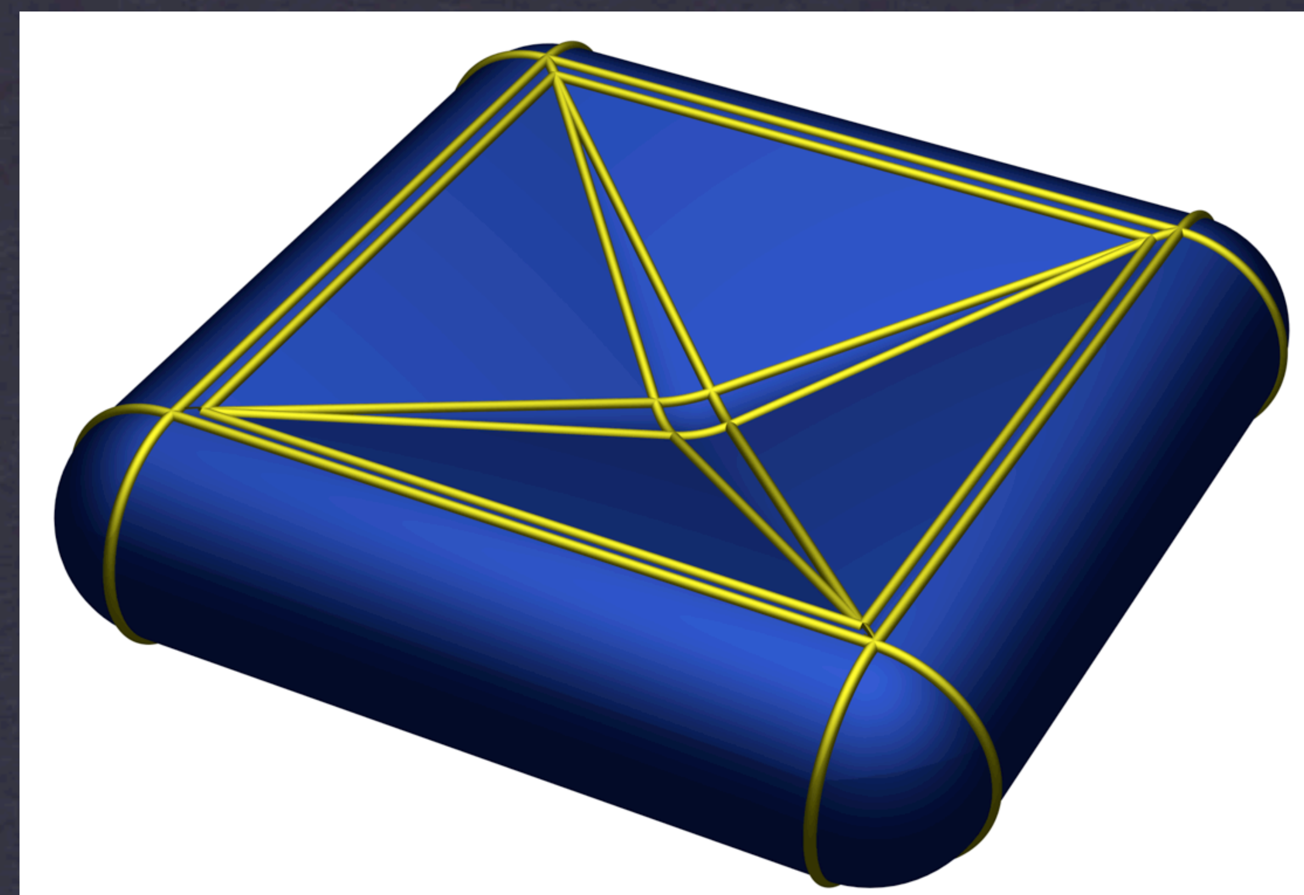
Tight Blend Definition

A *tight r-blend* of S is a tight hull of $S \circ_r$ relative to $S^c \circ_r$.

The boundary of a tight blend lies in $S \bullet_r \setminus S \circ_r$.

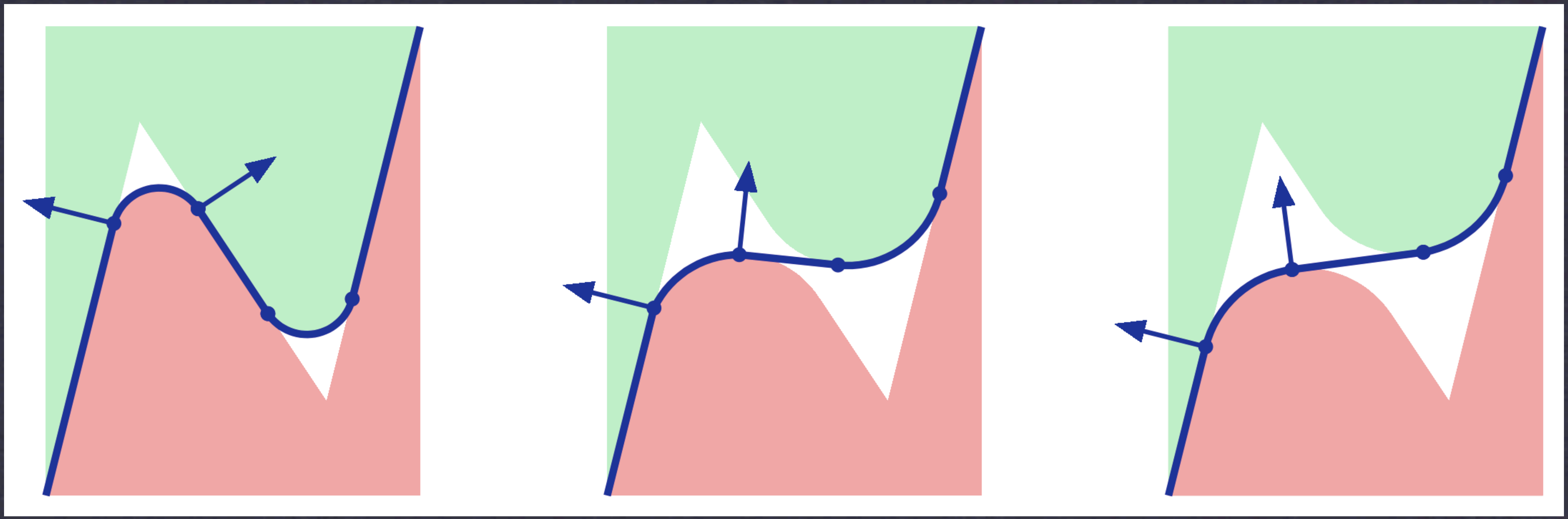


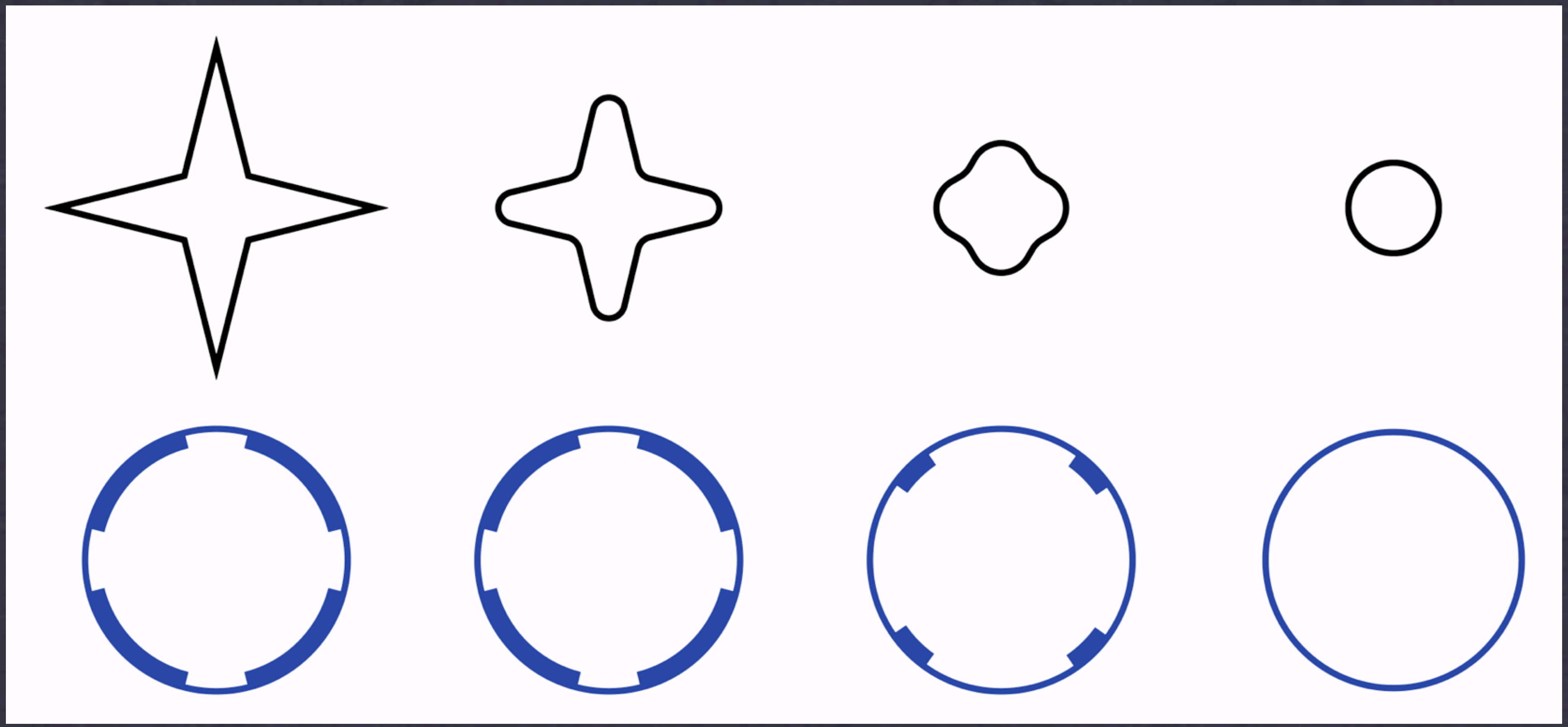




TWO AND THREE DIMENSIONS

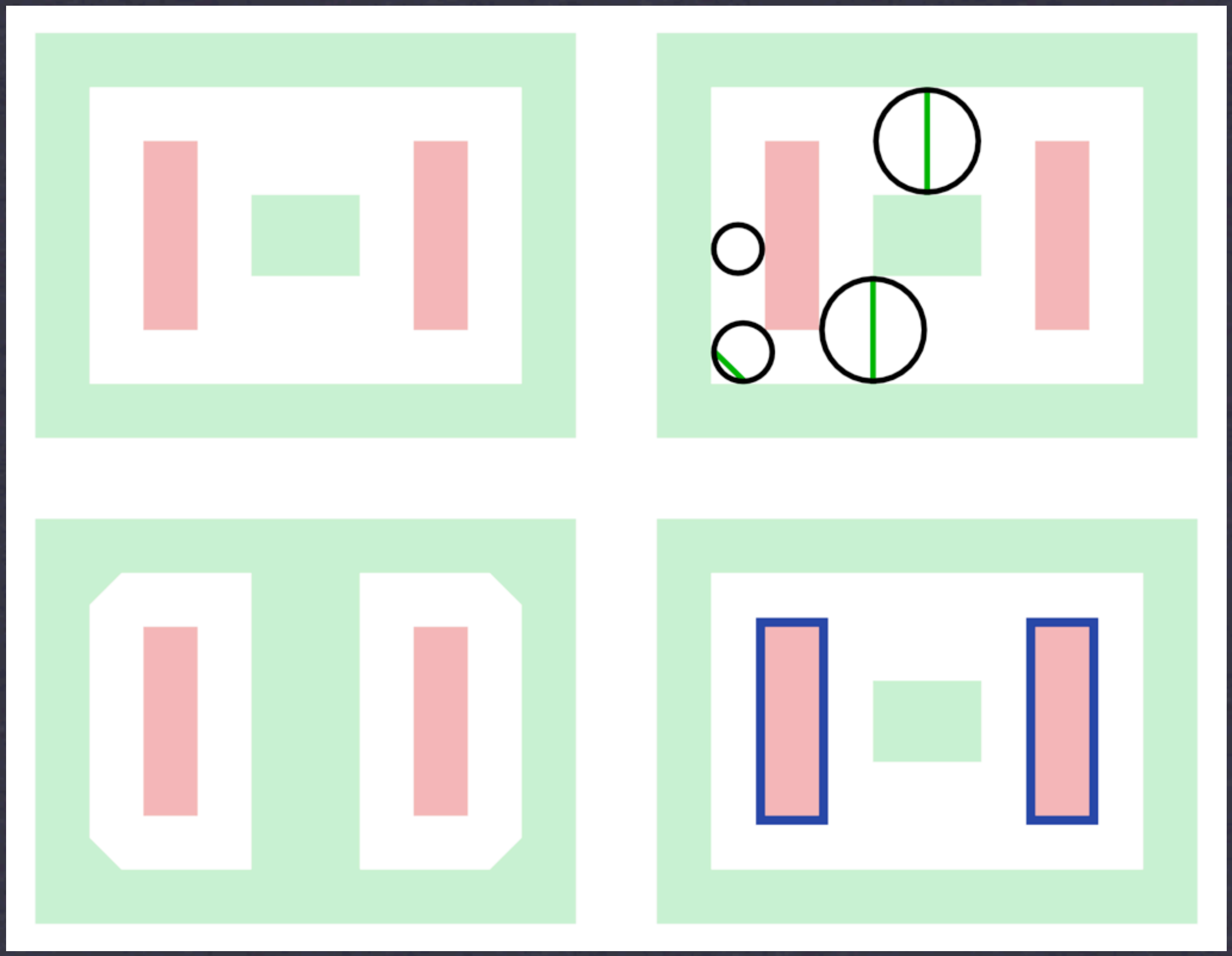
CONJECTURE: A BOUNDARY POINT LIES ON AN ARC OF RADIUS -1/R TO 1/R.

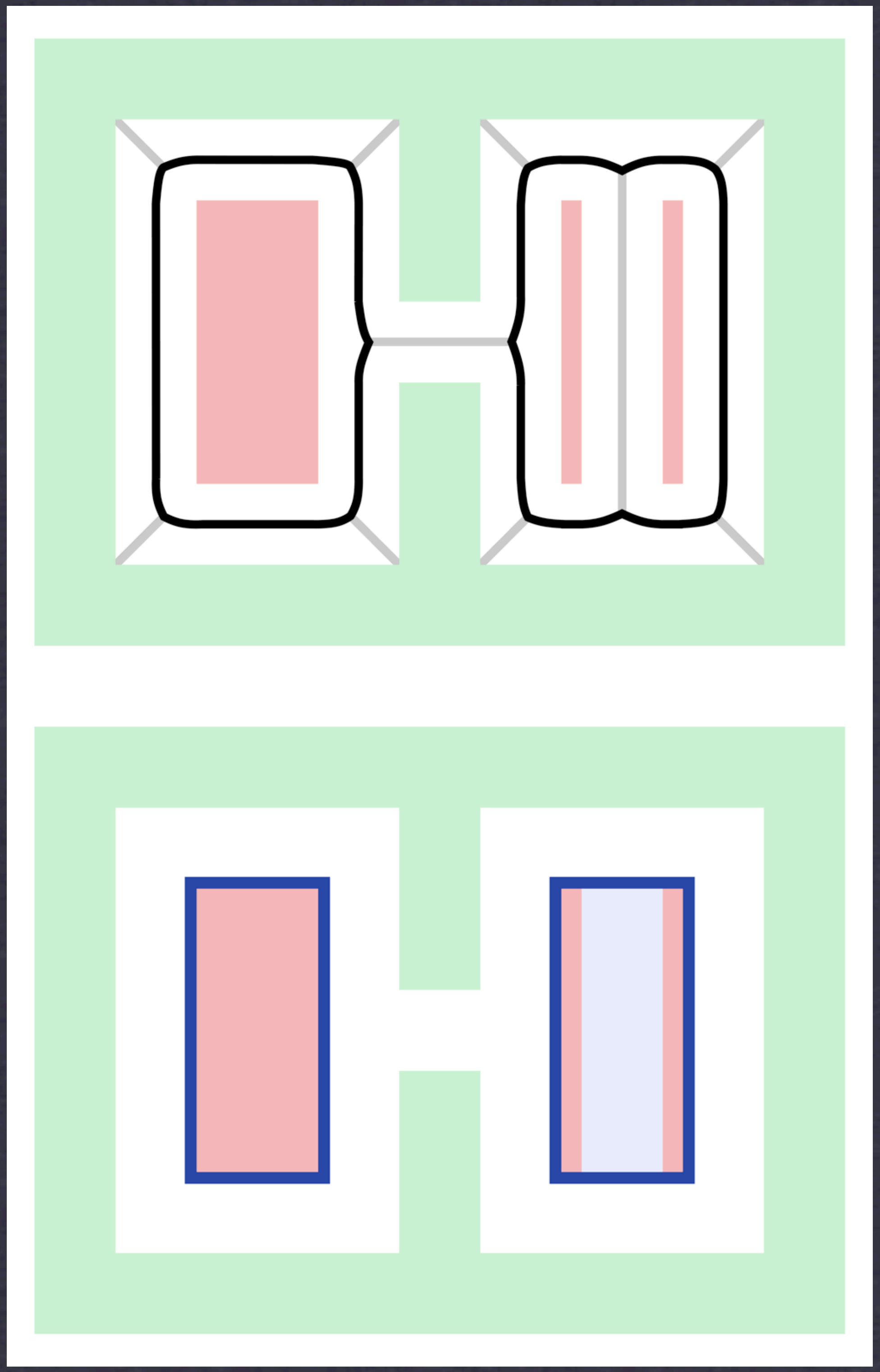




NORMAL FIELD SIMPLIFICATION

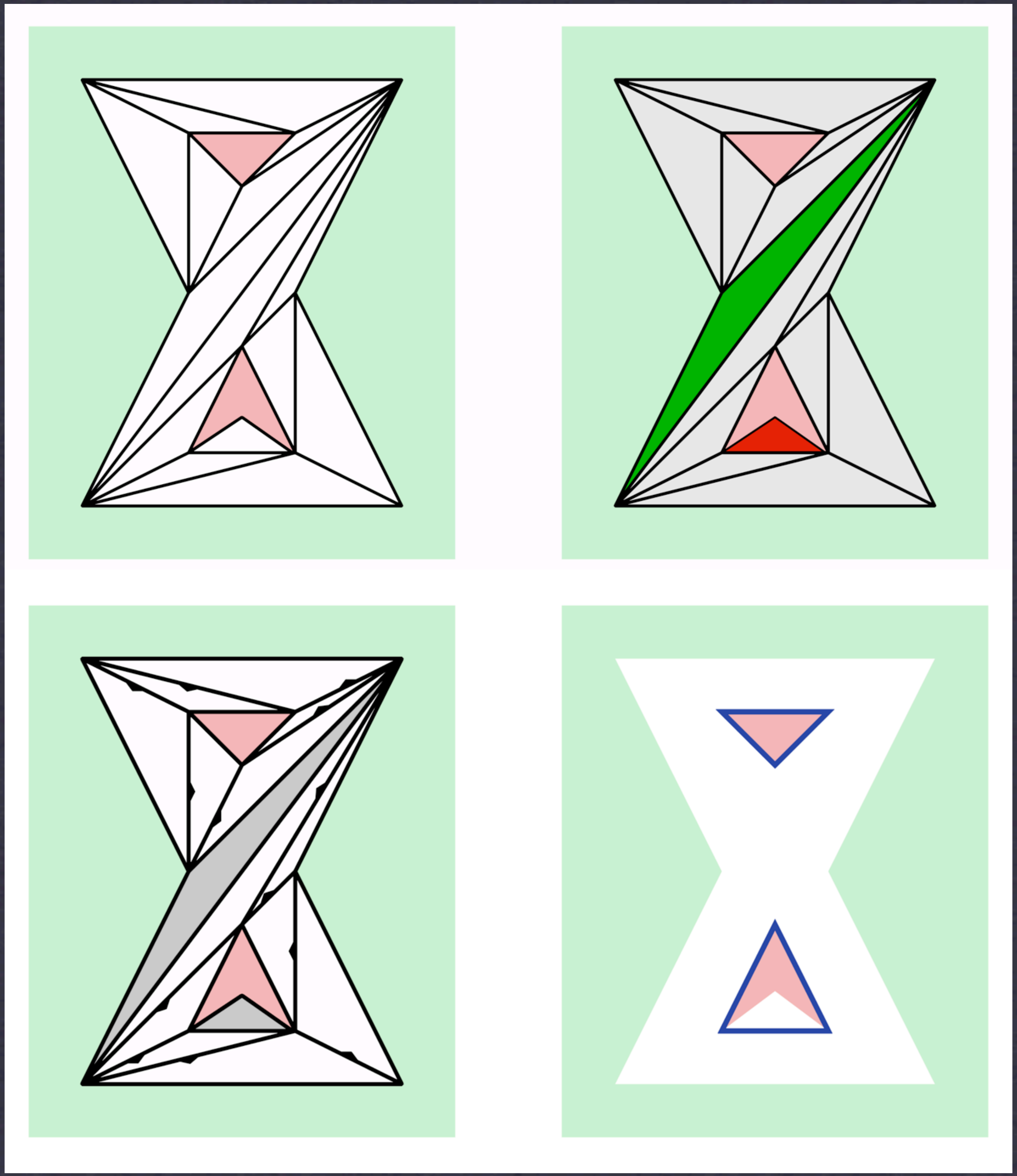
NORMAL CHANGE DECREASES AS OPENING RADIUS INCREASES WITH CONTINUOUS TOPOLOGY.





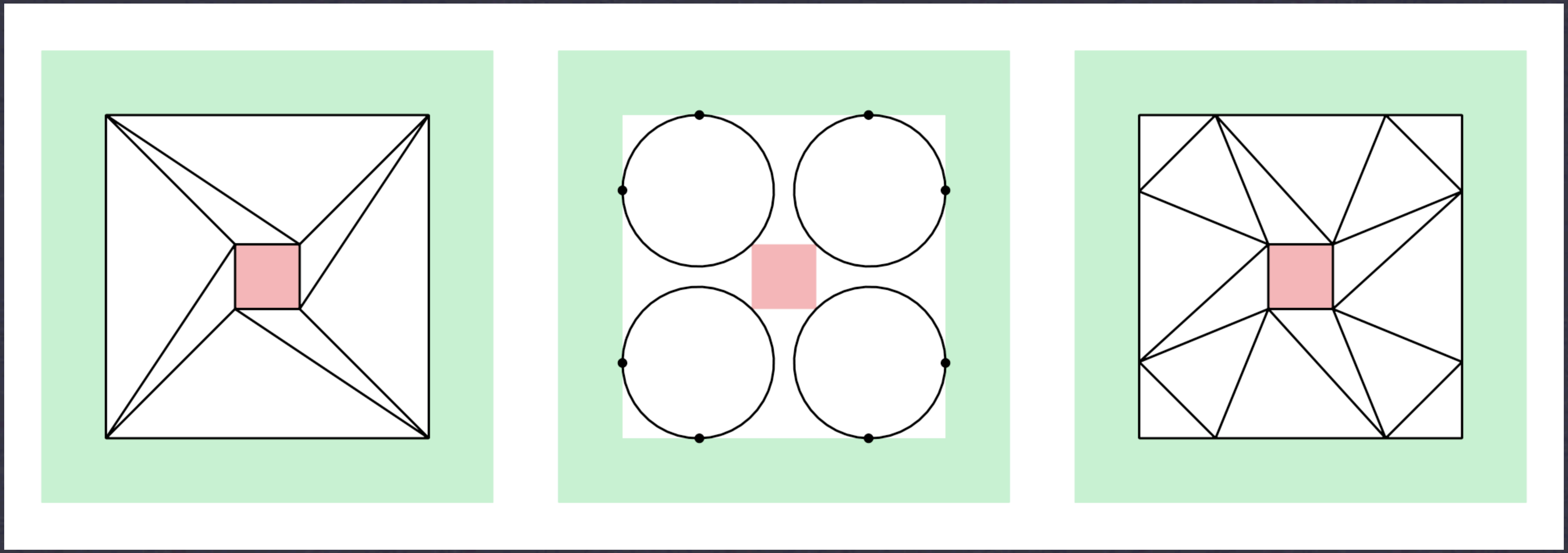
MEDIAL COVER

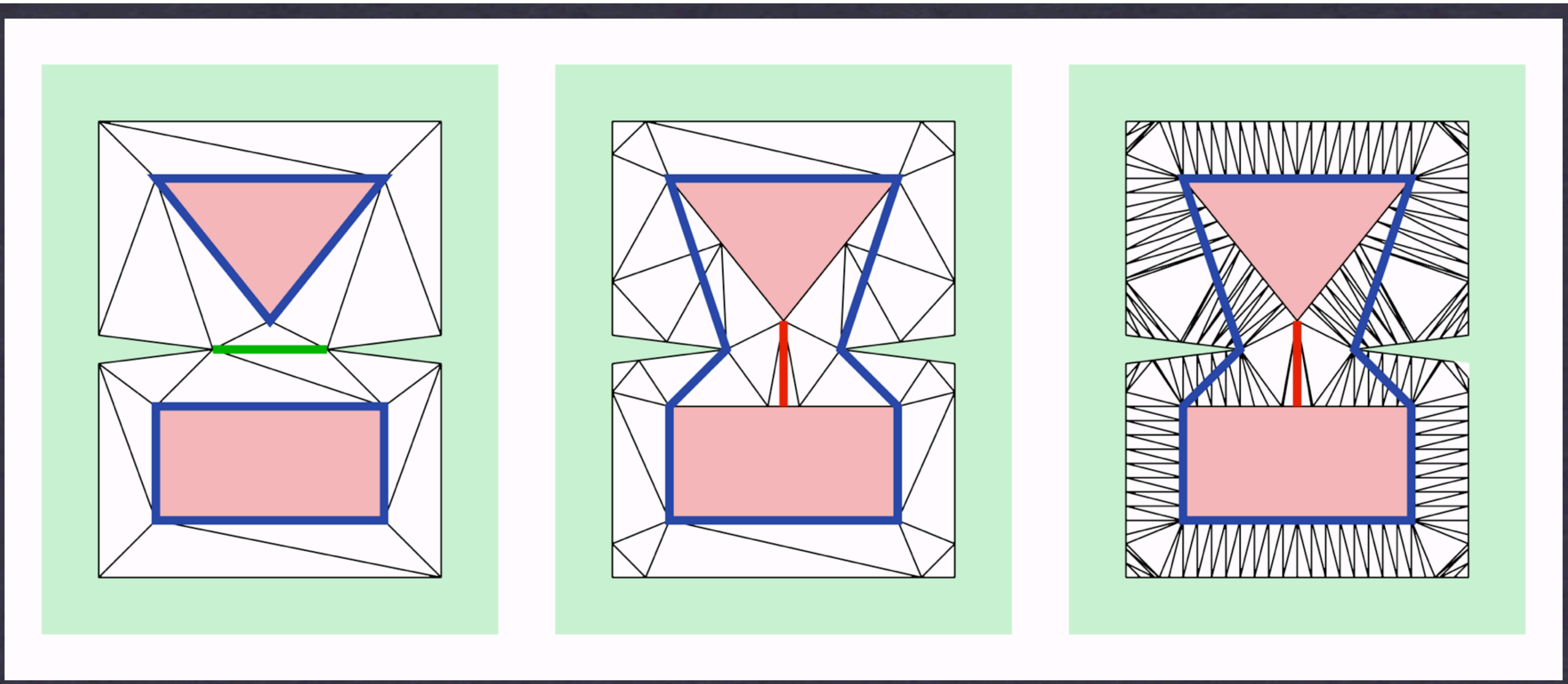
HULL BOUNDARY ISOTOPIC TO MEDIAL AXIS SUBSET EQUIDISTANT FROM R AND G



FUNNEL ALGORITHM

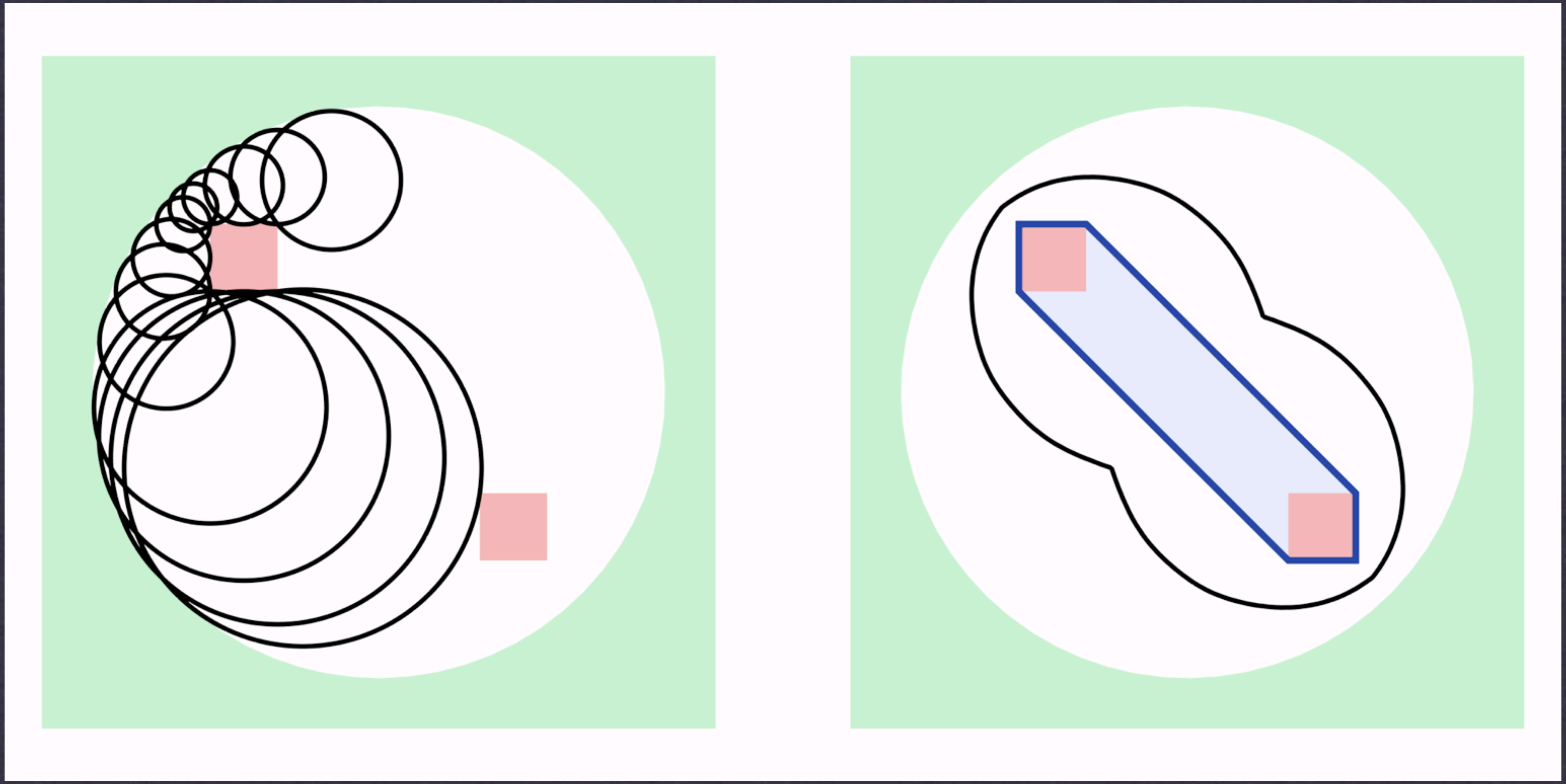
A TRIANGULATION COLORED BY VERTICES DEFINES ANNULI THAT DETERMINE PATHS.





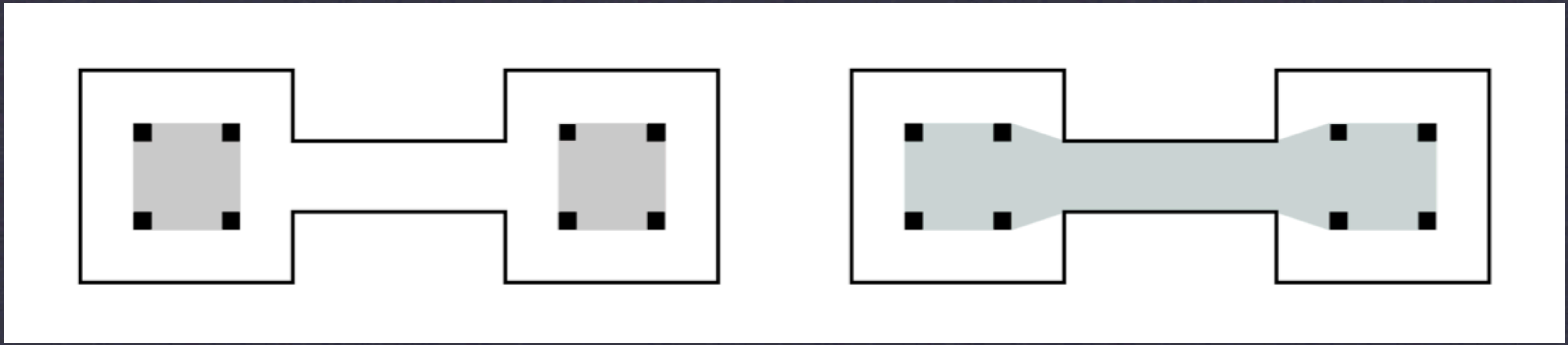
INSERTING CONTACT POINTS

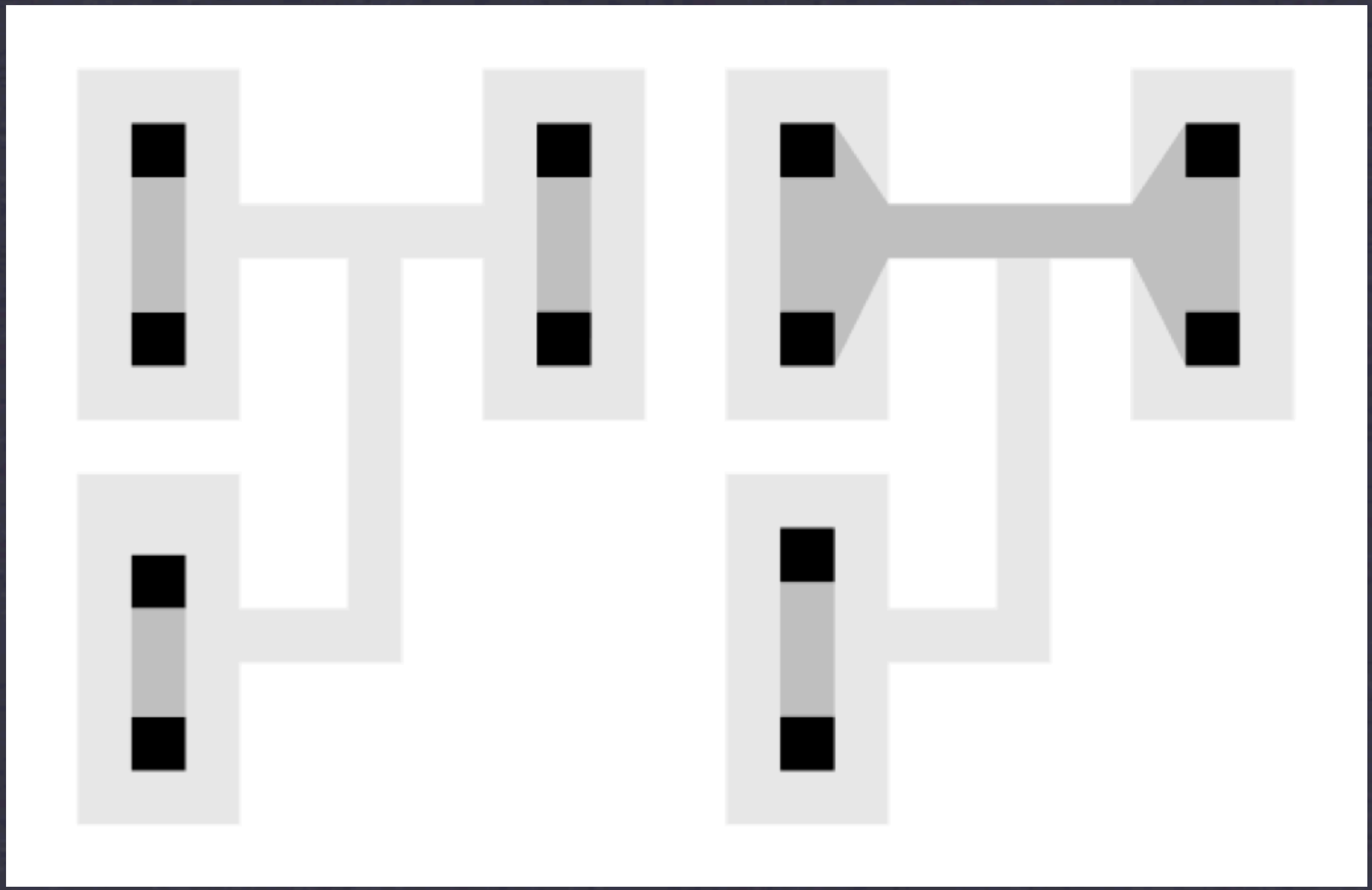
INSERTING POINTS WHERE BIFURCATION DISKS CONTACT EDGES MODIFIES TRIANGULATION.



MEDIAN ISOTOPY

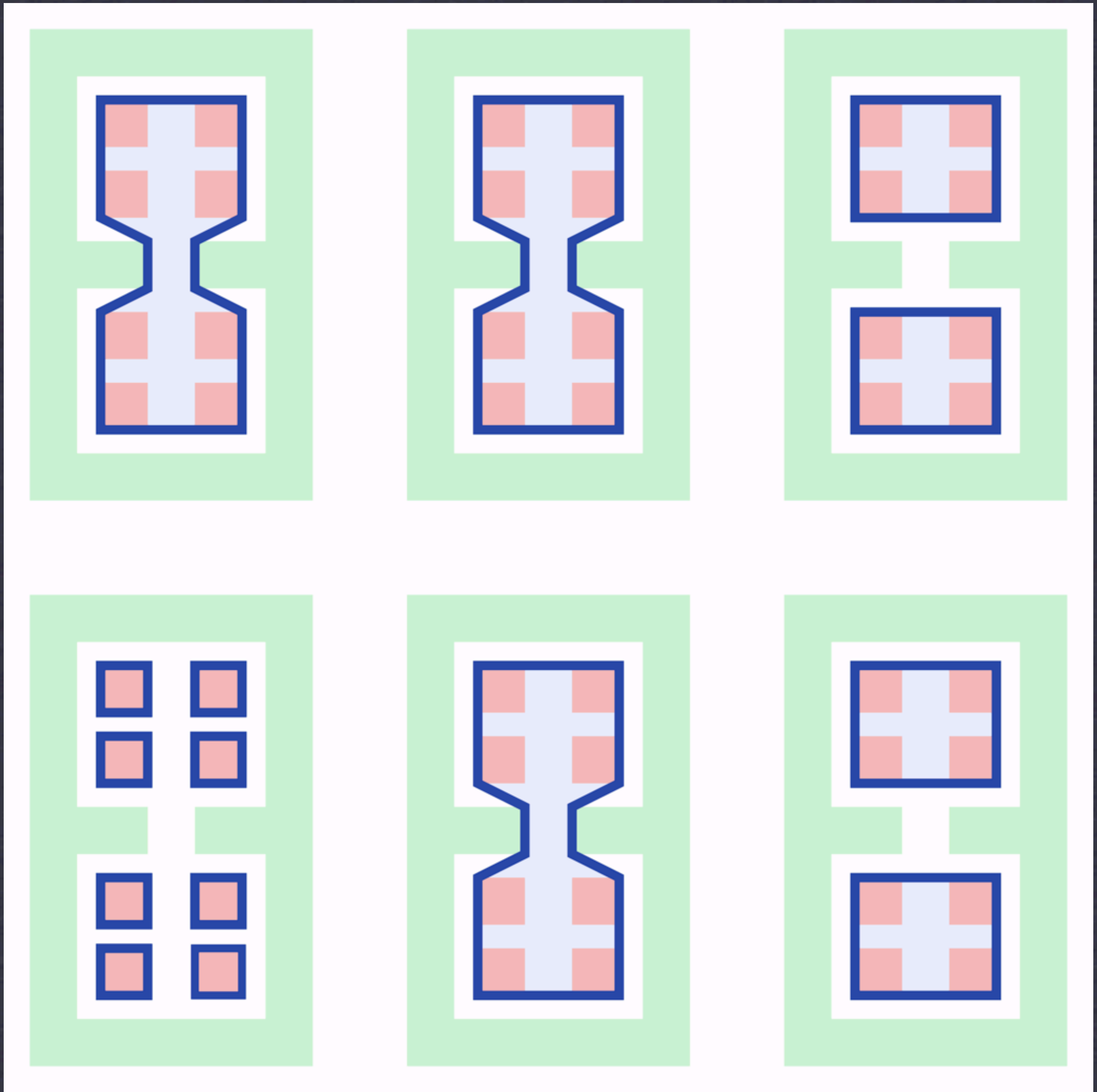
CENTERS OF MAXIMAL DISKS EQUIDISTANT FROM R AND G ARE ISOTOPIC TO BOUNDARY.





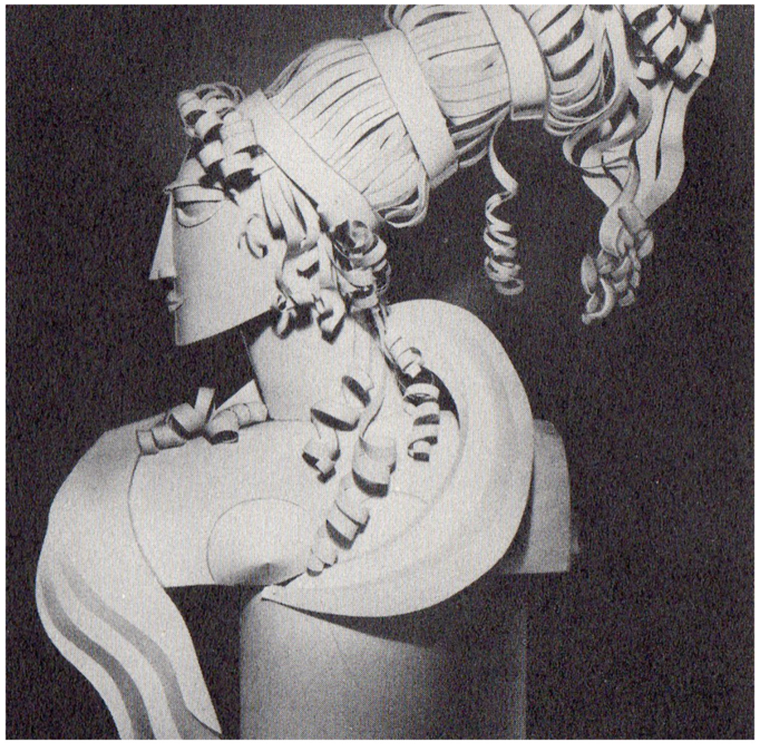
PROXIMITY

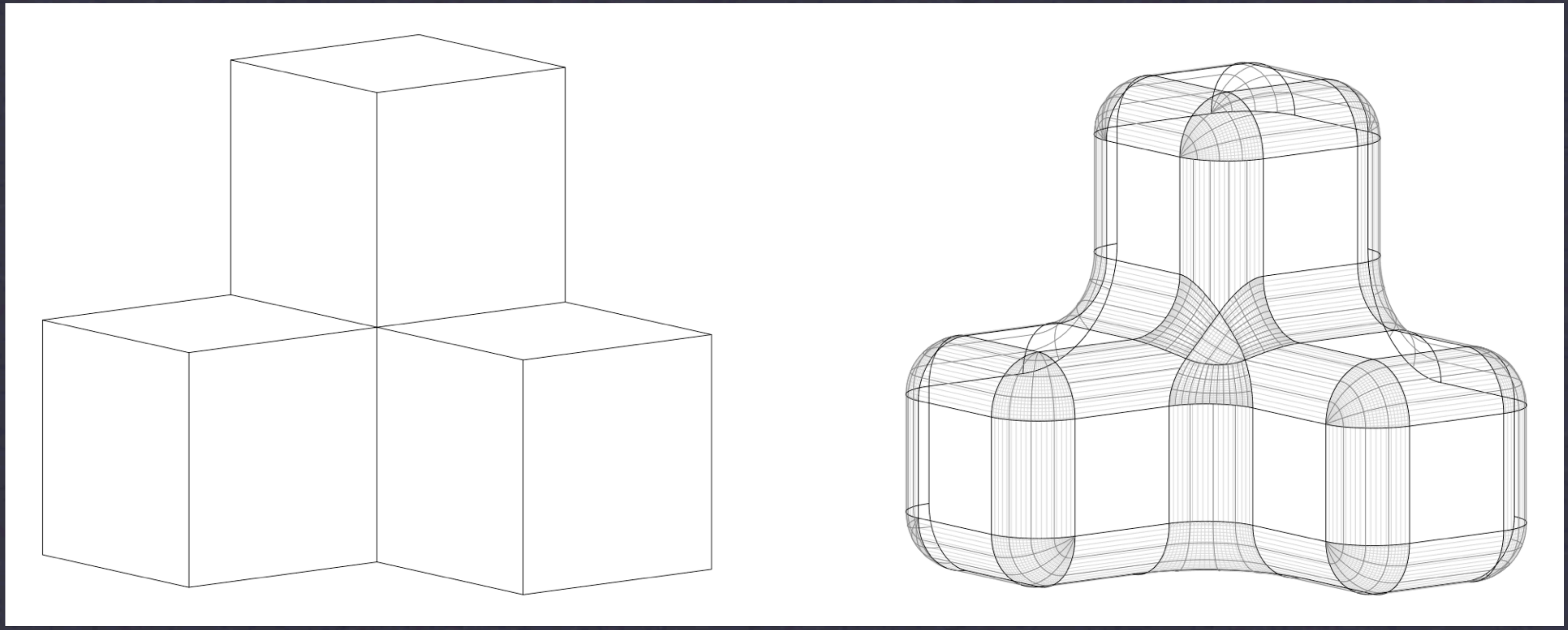
THE MEDIAL COVER SELECTIVELY COLLECTS NEARBY COMPONENTS.



SYMMETRY

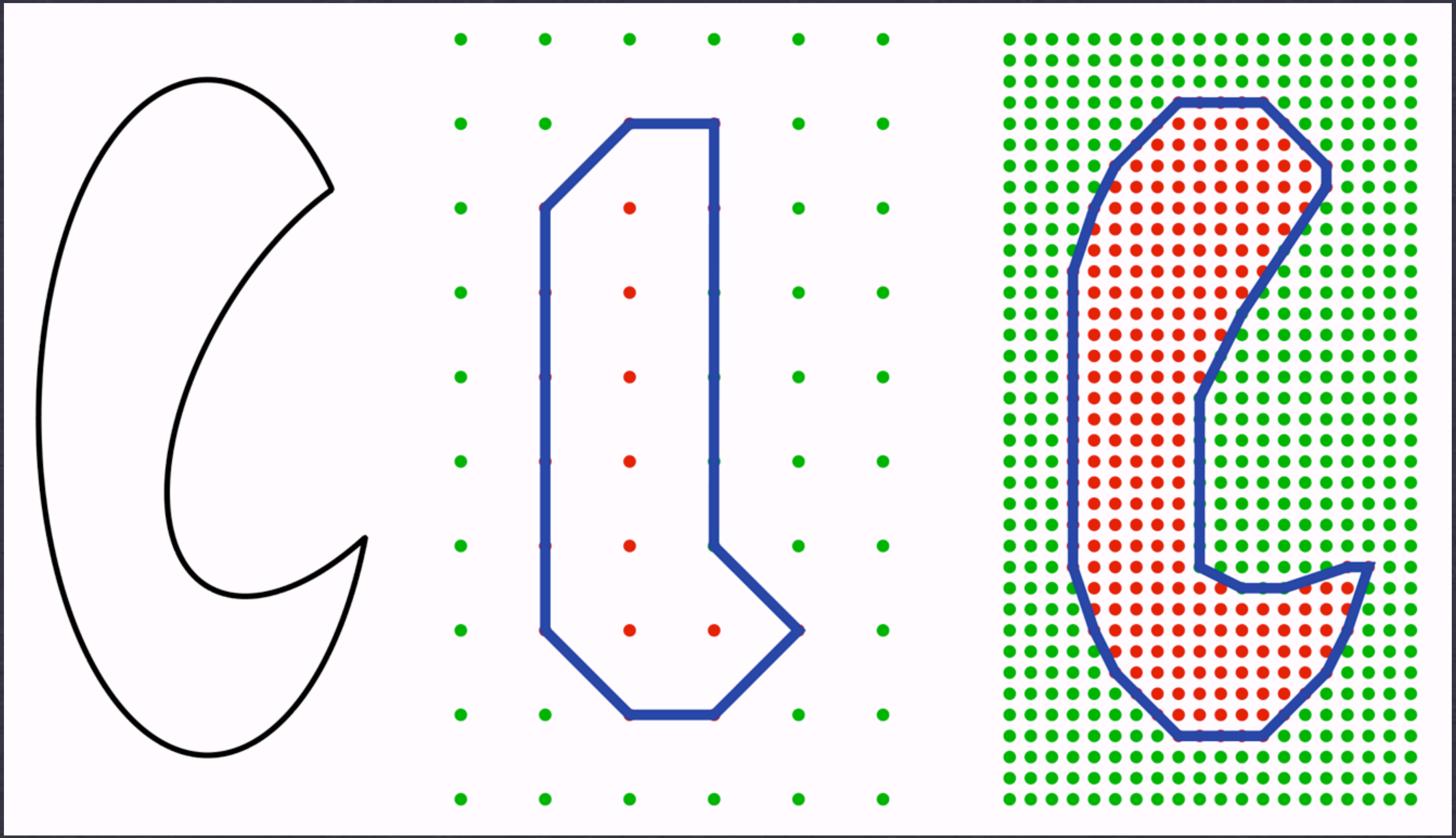
RCH(RIG) AND RCH(GIR) ARE DIFFERENT, WHILE TIGHT HULL AND MEDIAL COVER ARE NOT.





APPLICATIONS

SOLID MODEL BOUNDARY OPTIMIZATION USING TIGHTENING AND BLENDING

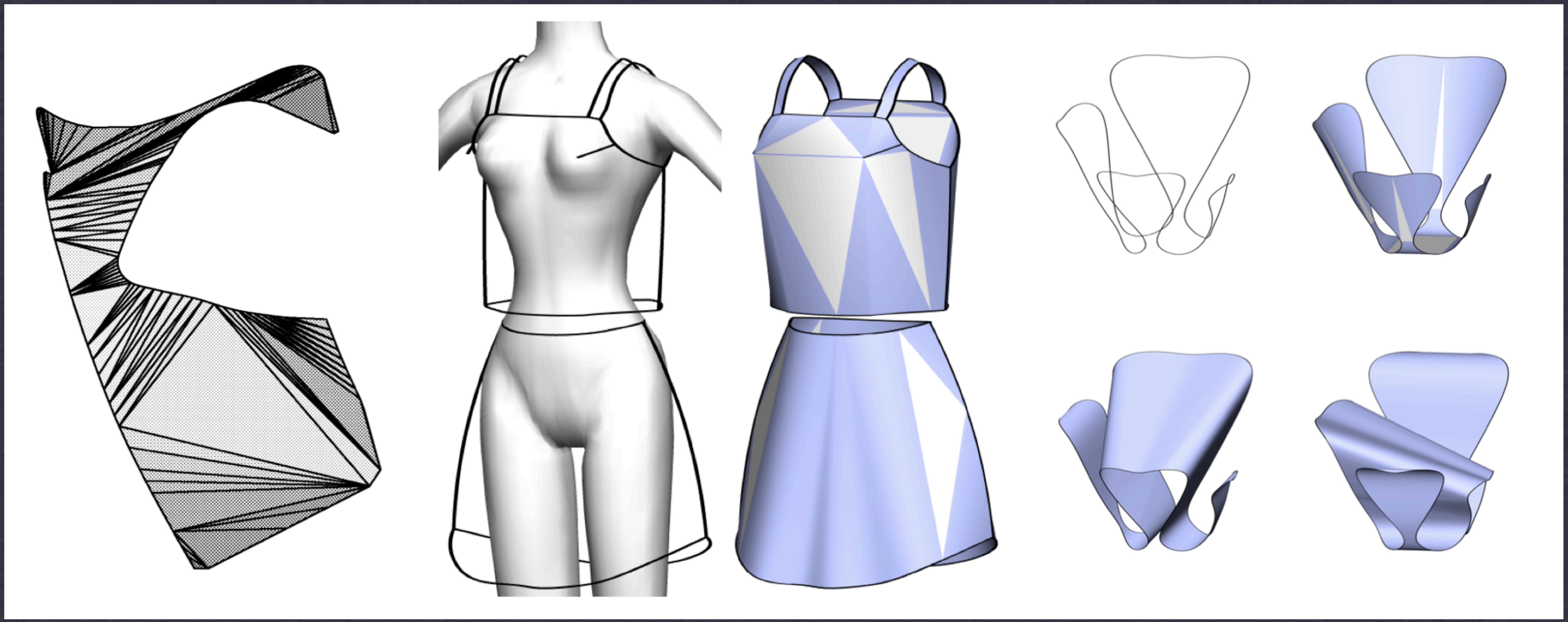


BOUNDARY ESTIMATION

TIGHT HULL SEPARATING RED AND GREEN SAMPLES CONVERGES WITH DENSITY

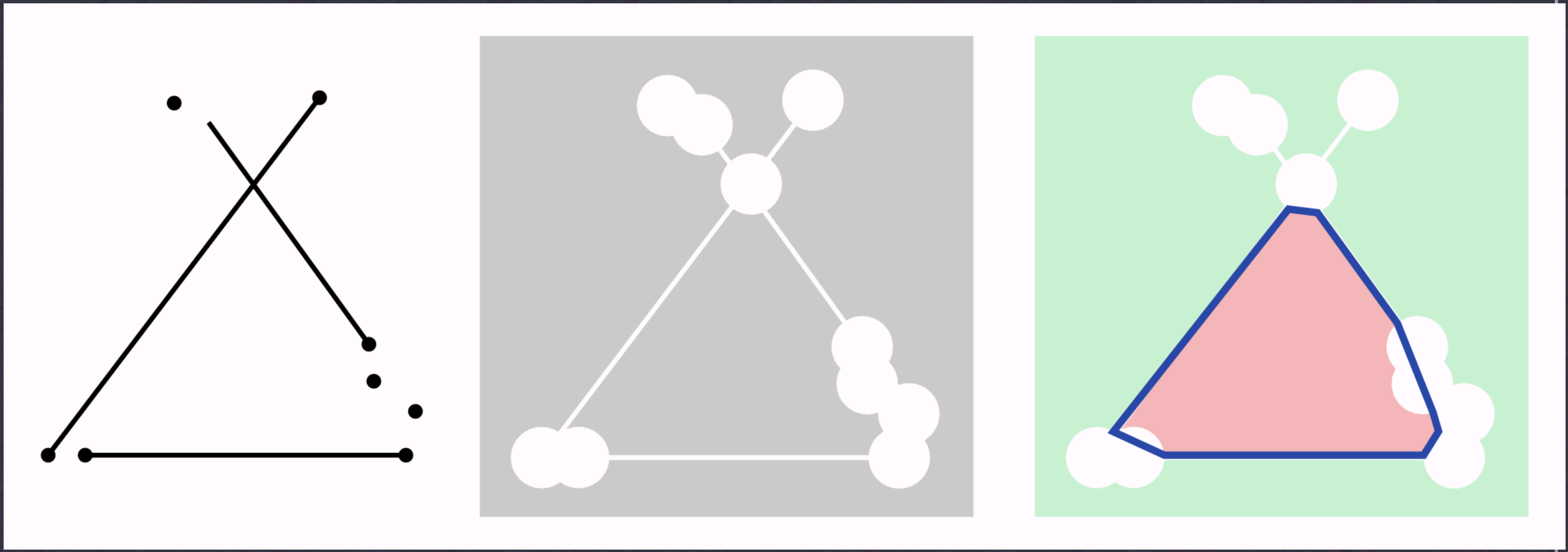
Biomedical Boundary Reconstruction

- Segment volumetric biomedical image data.
- Construct tight hull of interior relative to exterior.
- Geometric error is limited by intersample spacing.
- Conjecture: Normal field converges to input data.
- Accurate normals yield accurate shading.
- Tight hull accurately constructs model from samples.



DEVELOPABLE MANUFACTURE

UNSUPPORTED TIGHT HULL BOUNDARY MAY BE DEVELOPABLE, MADE OF FLEXIBLE SHEETS.



POLYGONAL REPAIR

CONSTRUCT THE TIGHT HULL IN THE UNION OF EDGES AND DILATED VERTICES.

Conclusion

- Fundamental problem:
  - Contain a subset of the input.
  - Exclude a subset of the complement.
  - Separate with an optimized boundary.
- Fundamental objectives:
  - Replace thin, sharp features with smooth, thick
  - Minimize boundary and simplify normal field
  - Produce output resembling the input
  - Handle input and complement symmetrically

Conclusion

- Mason: symmetrically regularize
- Tightening: bound mean curvature with flow
- Tight Hull: symmetric convex hull generalization
- Tight Blend: tight hull normal field simplification
- Medial Cover: medial hull and blend topology

Conclusion

- Tight hull of three-dimensional polygonal input might be constructed low-order polynomial time.
- Regularization may provide a round, thick output sampled with limited error and topological change.
- Work involving morphology, fairing, and convexity significantly develops aspects of solid modeling.